Differential Topology - Problem Set 5 (due: Dec 5)

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1. Poincare lemma for star shaped domains
Let $U \subset \mathbb{R}^n$ be open and star-shaped with respect to the origin. Suppose $\omega = \sum \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ is a closed k-form on U. Find an explicit formula for an antiderivative, i.e. explicitly find a $(k-1)$-form η on U that satisfies $d\eta = \omega$.

2. Moser's trick for volume forms
Let M be an oriented connected closed smooth manifold. Recall that a volume form on M is a positive smooth section of $\bigwedge^{\dim M} T^*M$. Now suppose ω_0, ω_1 are two volume forms. Prove that there exists a diffeomorphism ψ such that $\omega_1 = \psi^*\omega_0$ if and only if $\int_M \omega_0 = \int_M \omega_1$. (Hint: Writing $\omega_t = \omega_0 + td\lambda$ for some suitable λ, find a time-dependent vector field X_t such that $\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t = 0$, and consider its flow.)

3. Cohomology of genus g surface
Let Σ_g be a closed oriented surface of genus g. Use the Mayer-Vietoris theorem to compute the de Rham cohomology $H^*(\Sigma_g)$.

4. Lusternik-Schnirelman vanishing lemma for de Rham cohomology
Let M be an oriented smooth closed manifold. Recall that for any $U \subset M$ open, we have a restriction map $r_U : H^*(M) \to H^*(U)$ in de Rham cohomology. Now suppose that $U, V \subset M$ are open sets with $M = U \cup V$ and suppose that $[\omega], [\eta] \in H^*(M)$ are cohomology classes satisfying $r_U([\omega]) = 0$ and $r_V([\eta]) = 0$, respectively. Prove that $[\omega] \wedge [\eta] = 0$ in $H^*(M)$. (Hint: First show that given any compact subset $K \subset U$ one can find a representative of the cohomology class $[\omega]$ that vanishes pointwise on K, i.e. show that $[\omega] = [\tilde{\omega}]$ for some $\tilde{\omega} \in \Omega^*(M)$ that vanishes pointwise on K.)

Please write up (preferably using LaTeX) your solutions in a clear and concise way. Please hand in your work by the deadline (late submissions will not be accepted).