Differential Topology - Problem Set 4 (due: Nov 17)

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1. Normal bundle

Let $M^m \subset \mathbb{R}^n$ be a smooth embedded submanifold. Consider the normal bundle $NM := \{(p,v) \in M \times \mathbb{R}^n \,|\, v \perp T_pM\}$. Prove that NM is a smooth vectorbundle of rank $n-m$ and that $T\mathbb{R}^n|_M$ is isomorphic to $TM \oplus NM$.

2. Tubular neighborhoods and nearest neighbor projection

Let $M^m \subset \mathbb{R}^n$ be a smooth compact embedded submanifold, and consider the ε-neighborhood $M_\varepsilon := \{x \in \mathbb{R}^n \,|\, d(x,M) < \varepsilon\}$. For $\varepsilon > 0$ small enough, show that $(p,v) \mapsto p+v$ maps the ε-neighborhood of $M \times \{0\}$ in NM diffeomorphically onto M_ε. Moreover, considering the projection $\pi : M_\varepsilon \to M$ defined by $\pi(p+v) = p$, show that $\pi(p+v)$ is closer to $p+v$ than any other point of M.

3. Differential forms on S^1

Let $\omega \in \Omega(S^1)$. Prove that $\omega = df$ for some $f \in C^\infty(S^1)$ if and only if $\int_{S^1} \omega = 0$. (Hint: Let $\pi : \mathbb{R} \to S^1$, $\pi(t) = (\cos t, \sin t)$ and consider the pullback $\pi^*\omega$. Think about under what condition the function $g(t) := \int_0^t \pi^*\omega$ is 2π-periodic.)

4. Volume form on S^n

Let $\iota : S^n \to \mathbb{R}^{n+1}$ be the usual inclusion, and let $\omega \in \Omega^n(S^n)$ be given by

$$\omega = \iota^* \sum_{j=1}^{n+1} (-1)^{j-1} x^j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^{n+1}.$$

Show that ω is the standard volume form on S^n, i.e. show that $\omega_p(v_1, \ldots, v_n) = 1$ for any positively oriented orthonormal basis $v_1, \ldots, v_n \in T_pS^n$ at any $p \in S^n$.

Please write up (preferably using LaTeX) your solutions in a clear and concise way. Please hand in your work by the deadline (late submissions will not be accepted).