Differential Topology - Problem Set 3 (due: Oct 31)

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1. Pulling submanifolds apart
Let $X, Y \subset \mathbb{R}^n$ be compact submanifolds. Show that if $\dim(X)+\dim(Y) < n$, then for a.e. $s \in \mathbb{R}^n$ the shifted submanifold $X+s := \{x+s \,|\, x \in X\}$ is disjoint from Y. (Hint: What does transversality mean under our dimensional assumption?)

2. Degree of complex polynomial
Consider the complex polynomial $z^k + a_{k-1}z^{k-1} + \ldots a_0$. Show that this extends to a smooth map f from $S^2 = \mathbb{C} \cup \{\infty\}$ to itself, and compute its mapping degree.

3. Homotopies of maps between spheres
Show that every continuous map $f : S^m \to S^n$, where $m < n$, is homotopic to a constant map. (Hint: Show first that f can be approximated by a smooth map.)

4. Hopf invariant
Prove that the Hopf map $h : S^3 \to S^2$ (see problem set 1) is not homotopic to a constant map. (Hint: for two disjoint compact smooth oriented 1-dimensional manifolds $\Gamma_1, \Gamma_2 \subset \mathbb{R}^3$ one can define a linking number $\ell(\Gamma_1, \Gamma_2)$ as degree of the linking map $\Gamma_1 \times \Gamma_2 \to S^2$, $(x, y) \mapsto (x-y)/|x-y|$. Considering smooth maps $f : S^3 \to S^2$, show that $H(f) := \ell(f^{-1}(y), f^{-1}(z))$, where $y \neq z$ are regular values, is a well-defined homotopy invariant. Finally, compute $H(h)$ to conclude.)[1]

Please write up (preferably using LaTeX) your solutions in a clear and concise way. Please hand in your work by the deadline (late submissions will not be accepted).

[1] Historically, this invariant was discovered by Heinz Hopf in 1935. It was later shown that the homotopy group $\pi_3(S^2)$ is the infinite cyclic group $\mathbb{Z}$ generated by the Hopf map h.