Differential Topology - Problem Set 2 (due: Oct 13)

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1. Tangent space of graphs
Let $F : M \to N$ be a smooth map between smooth manifolds. Consider its graph

$$\Gamma := \{(x, y) \in M \times N \,|\, F(x) = y\}.$$

Show that Γ is a smooth manifold, and that for any $(x, y) \in \Gamma$ the tangent space

$$T_{(x,y)}\Gamma \subset T_xM \times T_yN$$

is equal to the graph of the linear map $dF_x : T_xM \to T_yN$.

2. Tangent bundles of spheres
For $n = 1, 3$ and 7 prove that TS^n is diffeomorphic to $S^n \times \mathbb{R}^n$. (Hint: Try to find n smooth vector fields that are linearly independent at every point. To do so it is useful to consider complex numbers, quaternions and octonions, respectively.)

3. Embedding the projective plane in $\mathbb{R}^4$
Prove that $\Phi : \mathbb{RP}^2 \to \mathbb{R}^4$, defined by

$$\Phi([x : y : z]) = \frac{1}{x^2 + y^2 + z^2}(xy, xz, yz, x^2 - y^2),$$

is an embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$. (Hint: It may be useful to figure out first at which points the map $\varphi : \mathbb{RP}^2 \to \mathbb{R}^3$, defined by

$$\varphi([x : y : z]) = \frac{1}{x^2 + y^2 + z^2}(xy, xz, yz),$$

fails to be an immersion.)

4. Flows and diffeomorphisms
Let M be a connected smooth manifold. Prove that the diffeomorphism group of M acts transitively on M, i.e. prove that for any $p, q \in M$ there exists a diffeomorphism $F : M \to M$ such that $F(p) = q$. (Hint: first show this for open balls in Euclidean space by constructing a suitable vectorfield and considering its flow.)

Please write up (preferably using LaTeX) your solutions in a clear and concise way. Please hand in your work by the deadline (late submissions will not be accepted).