Differential Topology - Problem Set 1 (due: Sep 29)

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1. Complex projective space
Consider $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$, where

$$(z_0, \ldots, z_n) \sim (w_0, \ldots, w_n) :\Leftrightarrow \exists \lambda \in \mathbb{C} \setminus \{0\} : (z_0, \ldots, z_n) = (\lambda w_0, \ldots, \lambda w_n).$$

- Show that $\mathbb{CP}^n$ is a smooth compact manifold of real dimension $2n$.
- Prove that $\mathbb{CP}^1$ is diffeomorphic to S^2.

2. Special orthogonal group
Consider $\mathrm{SO}(n) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \,:\, A^T A = I, \det(A) = 1\}$.

- Show that $\mathrm{SO}(n)$ is a smooth compact manifold of dimension $n(n-1)/2$.
- Prove that $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{RP}^3$.

3. Lens spaces
Given two coprime integers p, q consider $M_{p,q} := S^3/\sim$, where viewing $S^3 \subset \mathbb{C}^2$ the equivalence relation is generated by

$$(z_1, z_2) \sim (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2).$$

- Prove that $M_{p,q}$ is a smooth compact 3-manifold.
- Prove that $M_{p,q}$ and $M_{p,\tilde{q}}$ are diffeomorphic if $\tilde{q} \equiv \pm q^{\pm 1}$ (mod p).

4. Hopf fibration
Consider the Hopf map $F = \pi \circ \iota : S^3 \to \mathbb{CP}^1 \approx S^2$ defined as composition of the inclusion $\iota : S^3 \to \mathbb{C}^2 \setminus \{0\}$ and the canonical projection $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$.

- Find a formula for F in local coordinates, and describe the preimages of points.
- Compute the differential dF_p in local coordinates.

Please write up (preferably using LaTeX) your solutions in a clear and concise way. Please hand in your work by the deadline (late submissions will not be accepted).