

Harmonic functions

$$\boxed{\Delta u = 0}$$

(Laplace eqn)

1-dim: $u_{xx} = 0 \Rightarrow u(x) = A + Bx$

2-dim: $u_{xx} + u_{yy} = 0$

3-dim: $u_{xx} + u_{yy} + u_{zz} = 0$

Inhomogeneous version:

$$\boxed{\Delta u = f}$$

(Poisson eqn)

.) $u_t = \Delta u$ (or $u_{tt} = \Delta u$)

Steady (equilibrium) state:

$$u_t = 0 \Rightarrow \Delta u = 0.$$

much more interesting!

Notation φ in 2-dim

$$\Delta u = u_{xx} + u_{yy}$$

.) Electrostatics

$$\left\{ \begin{array}{l} \text{curl } \vec{E} = 0 \\ \text{div } \vec{E} = 4\pi\rho \end{array} \right. \quad \begin{array}{l} (1) \\ (2) \end{array}$$

↑ ↑
electric field electric charge

$$(1) \Rightarrow \vec{E} = -\text{grad } \phi \quad \text{for some potential } \phi$$

plug in (2). Get :

$$\underline{\underline{\Delta \phi}} = \text{div grad } \phi = -\text{div } \vec{E} = -\underline{\underline{4\pi\rho}}$$

(Poisson eqn with $f = -4\pi\rho$)

Complex analysis

$f(z)$ analytic function

$$z = x + iy \in \mathbb{C}, \quad f(z) = u(z) + i v(z)$$

↑ ↑
real part imaginary part

recall the Cauchy-Riemann eqns:

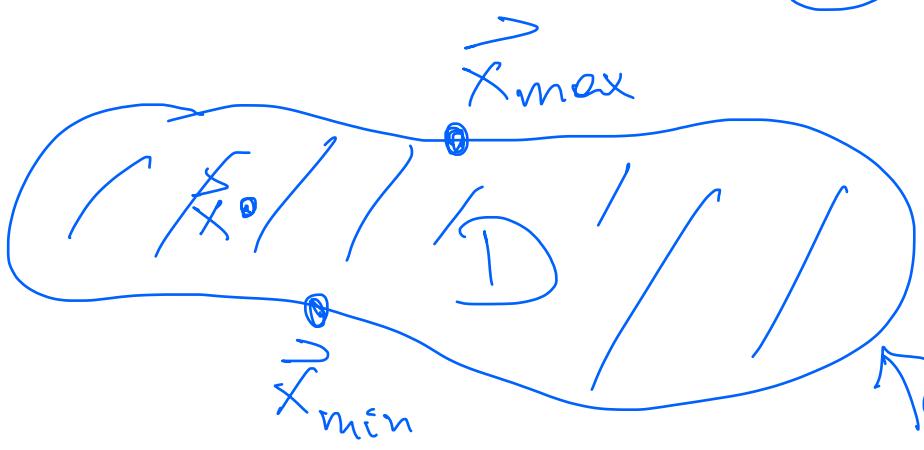
$$u_x = v_y, \quad u_y = -v_x$$

$$\Rightarrow u_{xx} = v_{yx} = v_{xy} = -u_{yy} \Rightarrow \Delta u = 0$$

Similarly, $\Delta v = 0$.

Maximum principle

D connected
bounded
open



$\bar{D} = D \cup \partial D$ boundary
of D

$u: D \rightarrow \mathbb{R}$ harmonic function,
continuous on \bar{D} .

$\Rightarrow \max_{\bar{D}} u, \min_{\bar{D}} u$ are attained on ∂D .

i.e. $\exists \vec{x}_{\min}, \vec{x}_{\max} \in \partial D$ s.t:

$$u(\vec{x}_{\min}) \leq u(\vec{x}) \leq u(\vec{x}_{\max}) \quad \forall \vec{x}.$$

Proof Let $\varepsilon > 0$. Consider $v(\vec{x}) = u(\vec{x}) + \varepsilon |\vec{x}|^2$

In 2-dim $\vec{x} = (x, y)$ (other dims,
similar)

$$\begin{aligned}\Delta v &= \underbrace{\Delta u}_{=0} + \varepsilon \Delta(x^2 + y^2) \\ &= (x^2 + y^2)_{xx} + (x^2 + y^2)_{yy} \\ &= 2+2 = 4\end{aligned}$$

$$\Rightarrow \Delta v = V_{xx} + V_{yy} = 4\varepsilon > 0 \text{ in } D.$$

2nd derivative test from calculus

$\Rightarrow v$ cannot have an interior
maximum in D .

Indeed, all eigenvalues of

the Hessian matrix $\begin{pmatrix} V_{xx} & V_{xy} \\ V_{xy} & V_{yy} \end{pmatrix}$

would be ≤ 0 . In particular,
taking the trace would

give $V_{xx} + V_{yy} \leq 0$.

$v : \bar{D} = D \cup \partial D \rightarrow \mathbb{R}$ continuous
 \nwarrow compact } $\Rightarrow \exists \text{maximum}$

$\Rightarrow \exists \vec{x}_0 \in \partial D : v(\vec{x}) \leq v(\vec{x}_0) \quad \forall \vec{x} \in \bar{D}$

$\Rightarrow u(\vec{x}) \leq v(\vec{x}) \leq v(\vec{x}_0) = u(\vec{x}_0) + \varepsilon |\vec{x}_0|^2$

$\leq \max_{\partial D} u + \varepsilon l^2$

$\forall \varepsilon > 0 \Rightarrow u(\vec{x}) \leq \max_{\partial D} u \quad \forall \vec{x} \in \bar{D}$

$\max_{\partial D} u = u(\vec{x}_{\max})$ for some $\vec{x}_{\max} \in \partial D$

$$\Rightarrow u(\vec{x}) \leq u(\vec{x}_{\max}) \quad \forall \vec{x} \in \bar{D}$$

Consider $-u \Rightarrow$ claim for min



Exer Show that the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } D \\ u = g & \text{on } \partial D \end{cases}$$

has at most one solution.

Let u_1, u_2 be two solutions, i.e.

$$\begin{cases} \Delta u_1 = f & \text{in } D \\ u_1 = g & \text{on } \partial D \end{cases}$$

$$\begin{cases} \Delta u_2 = f & \text{in } D \\ u_2 = g & \text{on } \partial D \end{cases}$$

Consider the difference $u := u_1 - u_2$

$$\Rightarrow \begin{cases} \Delta u = 0 \text{ in } D \\ u = 0 \text{ on } \partial D \end{cases}$$

Max princ $\Rightarrow \min u = \max u = 0$

$$\Rightarrow u \equiv 0 . \quad \square$$

Invariance in two dimensions

$\Delta u = u_{xx} + u_{yy}$ is invariant

under :

•) translations : $x' = x + a, y' = y + b$

$$\Rightarrow u_{x'x'} + u_{y'y'} = u_{xx} + u_{yy}$$

•) rotations : $x' = \underline{x \cos \alpha + y \sin \alpha}$

$$y' = -x \sin \alpha + y \cos \alpha$$

Exer Show $u_{x'x'} + u_{y'y'} = u_{xx} + u_{yy}$

$$\therefore u_x = u_{x'} \cos \alpha + u_{y'} (-\sin \alpha)$$

$$\therefore u_{xx} = u_{x'x'} \cos^2 \alpha + u_{x'y'} (-2 \sin \alpha \cos \alpha) \\ + u_{y'y'} \sin^2 \alpha$$

$$\therefore u_{yy} = u_{x'x'} \sin^2 \alpha + u_{x'y'} (2 \sin \alpha \cos \alpha) \\ + u_{y'y'} \cos^2 \alpha$$

$$\Rightarrow u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'}.$$

rotational invariance

\Rightarrow no preferred direction.

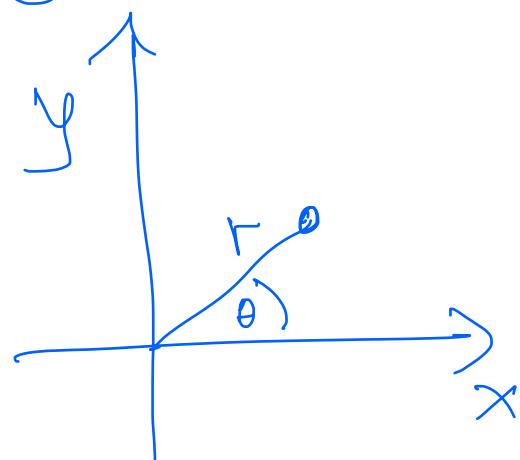
Hence, expect $\Delta_{2\text{-dim}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

becomes particularly simple in

Polar coordinates

$$x = r \cos \theta$$

$$y = r \sin \theta$$



Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix}$$

inverse $J^{-1} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos \theta & -\frac{\sin \theta}{r} \\ \sin \theta & \frac{\cos \theta}{r} \end{pmatrix}$

chain rule \Rightarrow

$$\left\{ \begin{aligned} \frac{\partial}{\partial x} &= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \end{aligned} \right.$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right)$$

$$= \cos^2 \theta \frac{\partial^2}{\partial r^2} - \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2}$$

$$+ \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial}{\partial r}$$

Similarly

$$\frac{\partial^2}{\partial y^2} = \sin^2 \theta \frac{\partial^2}{\partial r^2} + \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2}$$
$$- \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial}{\partial r}$$

$$\Rightarrow \boxed{\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}}$$

(2-dim Laplacian in polar coordinates)

Let's look for harmonic functions

depending only on r (but not on θ):

$$u_{rrr} + \frac{1}{r} u_r = 0$$

$$\Rightarrow (r u_r)_r = 0 \Rightarrow r u_r = c_1$$

$$\Rightarrow u = c_1 \log r + c_2$$

3-dim $\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

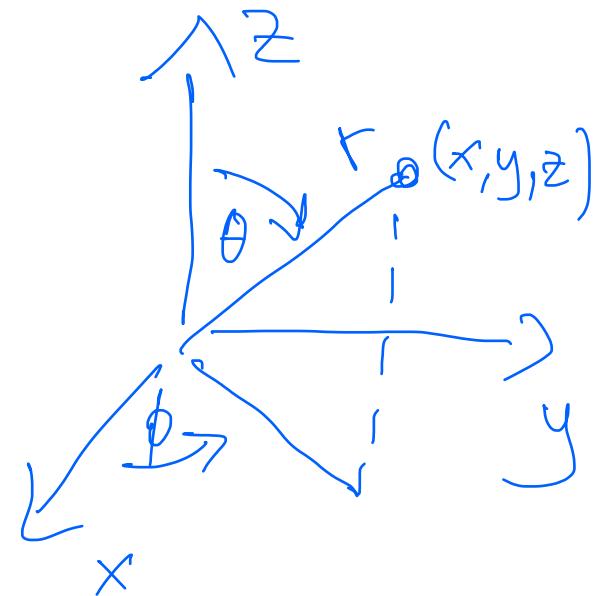
Again invariant under translations
and rotations.

Spherical coordinates

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$



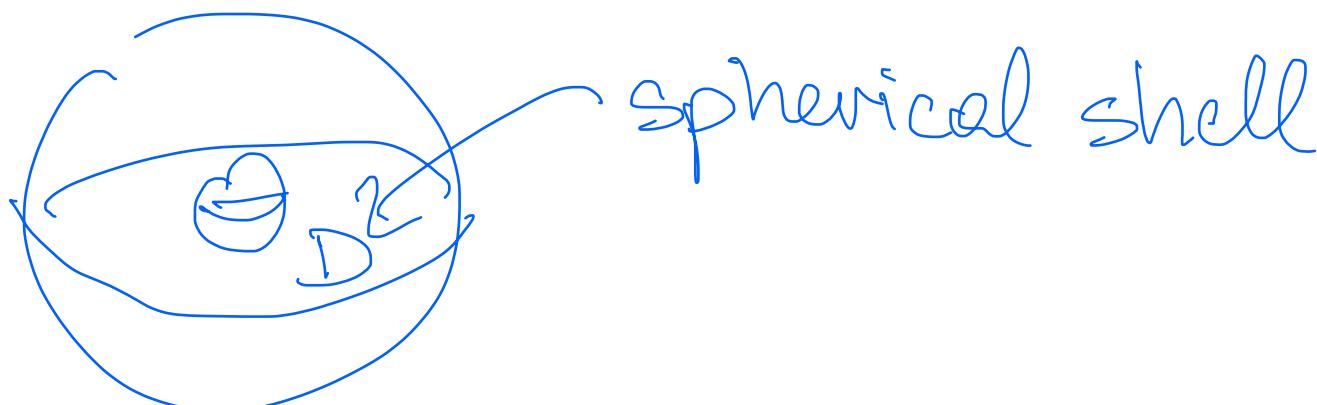
$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin^2 \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

(3d Laplacian in spherical coordinates)

Exer Let $D = \{(x, y, z) \in \mathbb{R}^3 : 1 < x^2 + y^2 + z^2 < 4\}$

Solve the Dirichlet problem

$$\left\{ \begin{array}{l} \Delta u = 0 \quad \text{in } D \\ u(x, y, z) = 0 \quad \text{if } x^2 + y^2 + z^2 = 1 \\ u(x, y, z) = \frac{1}{z} \quad \text{if } x^2 + y^2 + z^2 = 4. \end{array} \right.$$



At most 1 solution by maximum principle. Look for radial solution $u = u(r)$.

$$u_{rr} + \frac{2}{r} u_r = 0$$

$$\Rightarrow (r^2 u_r)_r = 0 \Rightarrow r^2 u_r = C_1$$

$$\Rightarrow u_r = \frac{C_1}{r^2} \Rightarrow u = -\frac{C_1}{r} + C_2$$

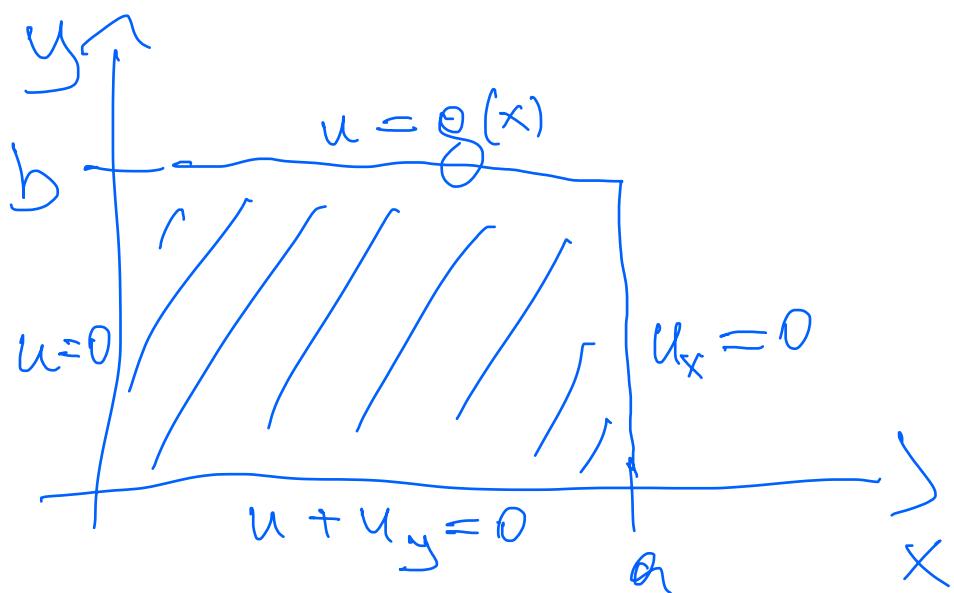
$$r=1 : \quad 0 = -c_1 + c_2 \Rightarrow c_2 = c_1$$

$$r=2 : \quad \frac{1}{2} = -\frac{c_1}{2} + c_1 = \frac{c_1}{2}$$
$$\Rightarrow c_1 = c_2 = 1$$

$$\Rightarrow u(r) = 1 - \frac{1}{r}$$

Exer Suppose the domain is the rectangle $D = \{0 < x < a, 0 < y < b\}$

Solve $\Delta u = 0$ in D with the boundary conditions:



Separation of variables method 2

$$u(x, y) = X(x)Y(y)$$

$$\Rightarrow X''Y + XY'' = 0$$

$$\Rightarrow \frac{Y''}{Y} = -\frac{X''}{X} = \lambda \text{ constant}$$

$$\left. \begin{array}{l} X'' + \lambda X = 0 \\ X(0) = 0, X'(a) = 0 \end{array} \right\} \Rightarrow X_n(x) = \sin(\beta_n x)$$

where

$$\beta_n^2 = \lambda_n = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{a^2}$$

$$\left. \begin{array}{l} Y'' - \lambda Y = 0 \\ Y'(0) + Y(0) = 0 \end{array} \right\} \Rightarrow Y_n(y) = \beta_n \cosh \beta_n y - \sinh \beta_n y$$

$$\Rightarrow u(x, y) = \sum_{n=0}^{\infty} A_n (\beta_n \cosh \beta_n b - \sinh \beta_n b) \cdot \sin(\beta_n x)$$

Finally, unknown. B.C.

$$g(x) = \sum_{n=0}^{\infty} A_n (\beta_n \cosh \beta_n b - \sinh \beta_n b) \sin(\beta_n x)$$

Fourier coeffs

$$\Rightarrow A_n = \frac{2}{a (\beta_n \cosh \beta_n b - \sinh \beta_n b)} \int_0^a g(x) \sin(\beta_n x) dx$$