The wave equation

Simplifying assumptions:

- 1-dimensional
- no boundary $(-\infty < x < \infty)$

$$\boxed{u_{tt} = c^2 u_{xx} \quad \text{for} \quad -\infty < x < \infty} \quad (*)$$

To solve it, we observe

$$\begin{aligned} 0 &= u_{tt} - c^2 u_{xx} \\ &= (\partial_t^2 - c^2 \partial_x^2) u \\ &= (\partial_t - c\partial_x) \underbrace{(\partial_t + c\partial_x) u}_{=:v} = (\partial_t - c\partial_x) v \\ &= v_t - c v_x \end{aligned}$$

$$(*) \Longleftrightarrow \begin{cases} v = u_t + c u_x \\ v_t - c v_x = 0 \end{cases}$$

By 1st week $v_t - c v_x = 0$ has the general solution

$$v(x,t) \overset{?}{=} h(x+ct)$$

Indeed:

$$\left.\begin{aligned} v_t &= c\,h'(x+ct) \\ v_x &= h'(x+ct) \end{aligned}\right\} \Rightarrow v_t = c v_x.$$

Hence, must solve

$$u_t + c u_x = h(x+ct)$$

Strategy: i) find special solution of inhomogeneous eqn.

ii) find general sol. of homog. eqn.

iii) put it together.

i) Note that $u(x,t) = f(x+ct)$,

where $f'(s) = h(s)/2c$,

is a solution.

Indeed, $u_t + cu_x = cf' + cf' = 2cf' = h$.

ii) $u_t + cu_x = 0$ has the general solution $u(x,t) = g(x-ct)$

iii) Thus, the general solution of $u_t + cu_x = h(x+ct)$

is $\boxed{u(x,t) = f(x+ct) + g(x-ct)}$

<u>Exer</u> Solve the initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = e^x, \quad u_t(x,0) = \sin x \end{cases}$$

General solution of $u_{tt} = c^2 u_{xx}$

is $u(x,t) = f(x+ct) + g(x-ct)$

Determine f & g from initial conditions:

$$\begin{cases} f(x) + g(x) = e^x \\ cf'(x) - cg'(x) = \sin x \end{cases}$$

$$\Rightarrow \begin{cases} f' + g' = e^x \\ f' - g' = \frac{1}{c}\sin x \end{cases}$$

$$\Rightarrow \begin{cases} f' = \frac{1}{2}\left(e^{x} + \frac{1}{c}\sin x\right) \\ g' = \frac{1}{2}\left(e^{x} - \frac{1}{c}\sin x\right) \end{cases}$$

$$\Rightarrow \begin{cases} f(x) = \frac{1}{2}\left(e^{x} - \frac{1}{c}\cos x\right) + C_1 \\ g(x) = \frac{1}{2}\left(e^{x} + \frac{1}{c}\cos x\right) + C_2 \end{cases}$$

$$\Rightarrow u(x,t) = \underbrace{\frac{1}{2}\left(e^{x+ct} - \frac{1}{c}\cos(x+ct)\right)}$$

$$\underbrace{+ \frac{1}{2}\left(e^{x-ct} + \frac{1}{c}\cos(x-ct)\right)}$$

$$+ \underbrace{C_1 + C_2}_{=0}$$

Initial value problem

$$\boxed{\begin{aligned} &u_{tt} = c^2 u_{xx} \\ &u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \end{aligned}}$$

Similarly as before, we have

$$u(x,t) = f(x+ct) + g(x-ct)$$

$$\left.\begin{aligned} \phi' &= f' + g' \\ \frac{1}{c}\psi &= f' - g' \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} f' &= \frac{1}{2}\left(\phi' + \frac{\psi}{c}\right) \\ g' &= \frac{1}{2}\left(\phi' - \frac{\psi}{c}\right) \end{aligned}\right.$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\Big(\phi(x+ct) + \phi(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds}$$

<u>Exer</u> Use this formula to solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) \equiv 0, \quad u_t(x,0) = \cos x \end{cases}$$

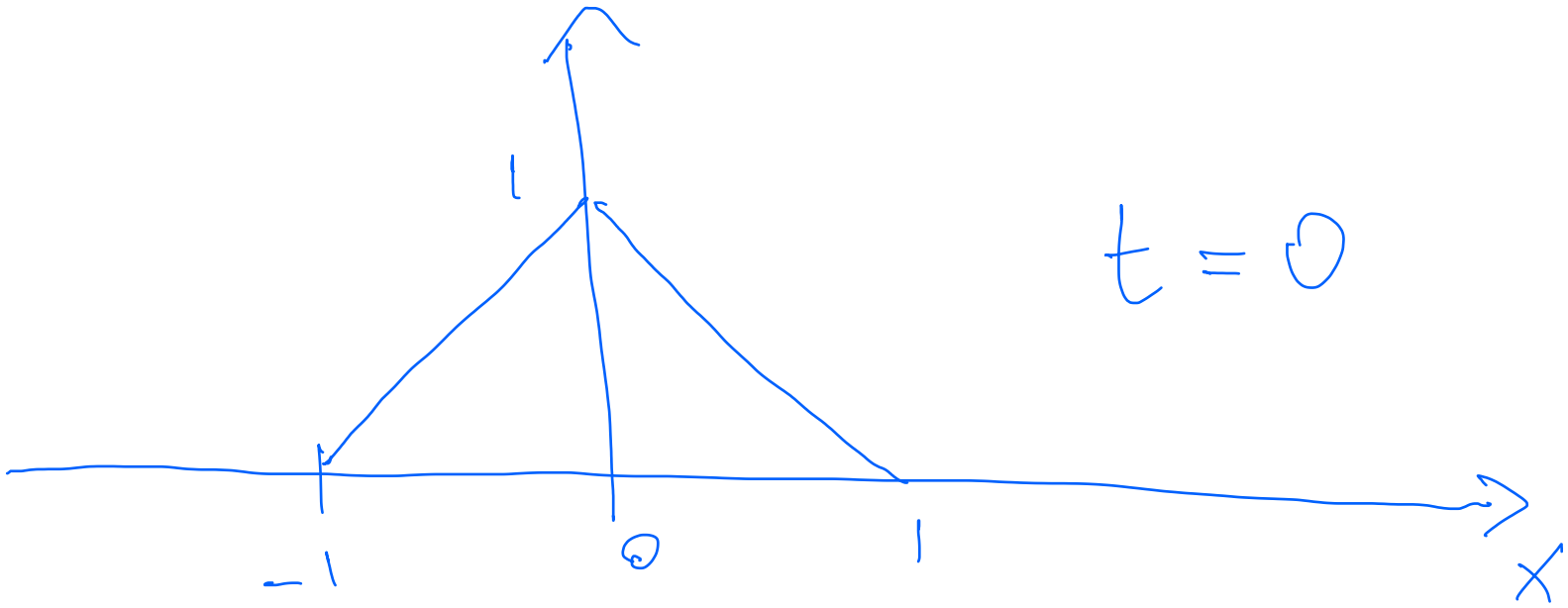
<u>Solution</u> $\phi = 0$, $\psi(s) = \cos(s)$

$$\Rightarrow u(x,t) = 0 + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos(s)\,ds$$

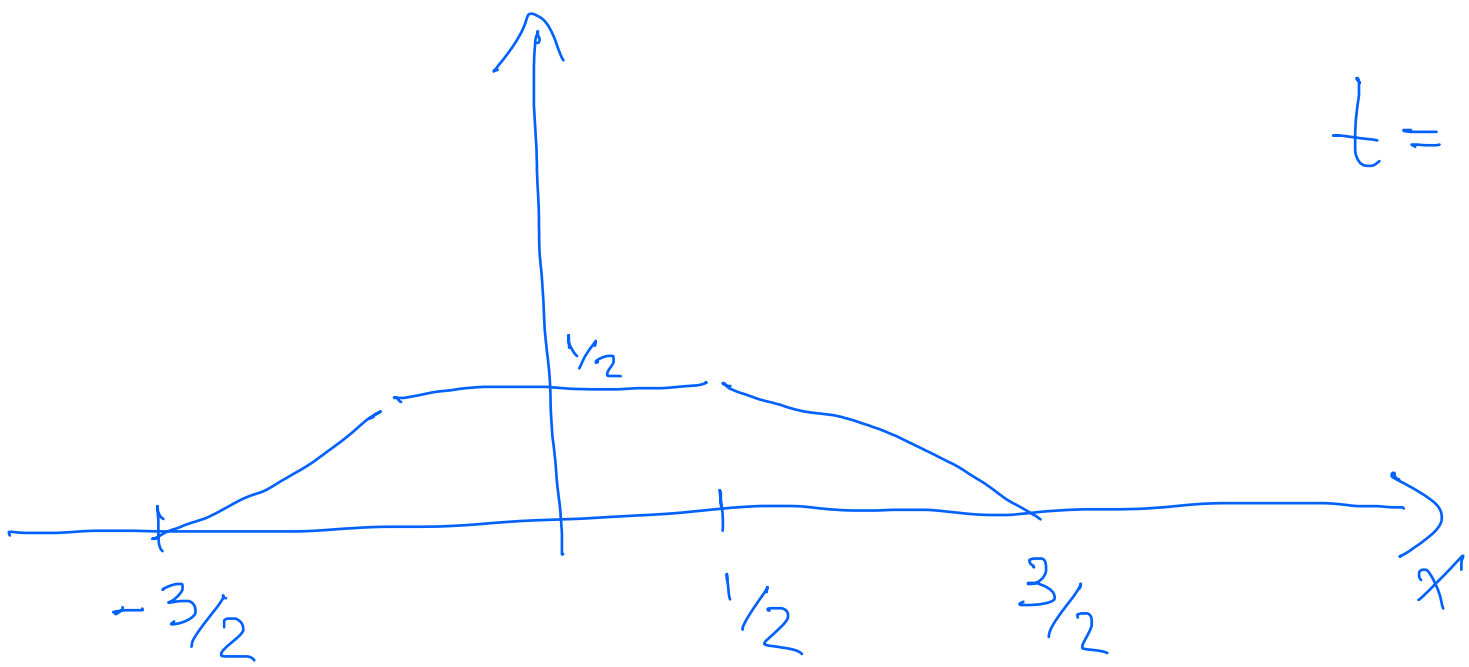
$$= \underline{\frac{1}{2c}\left(\sin(x+ct) - \sin(x-ct)\right)}$$

Example: The plucked string

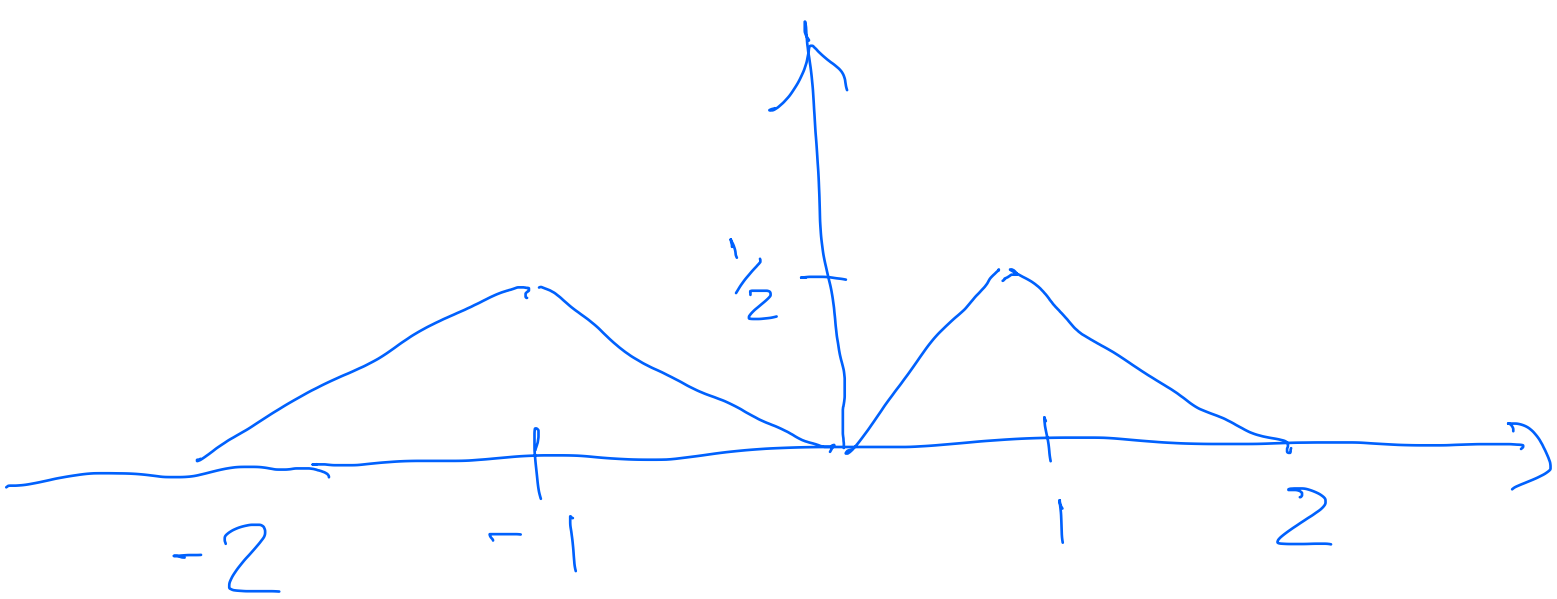
$c = 1$



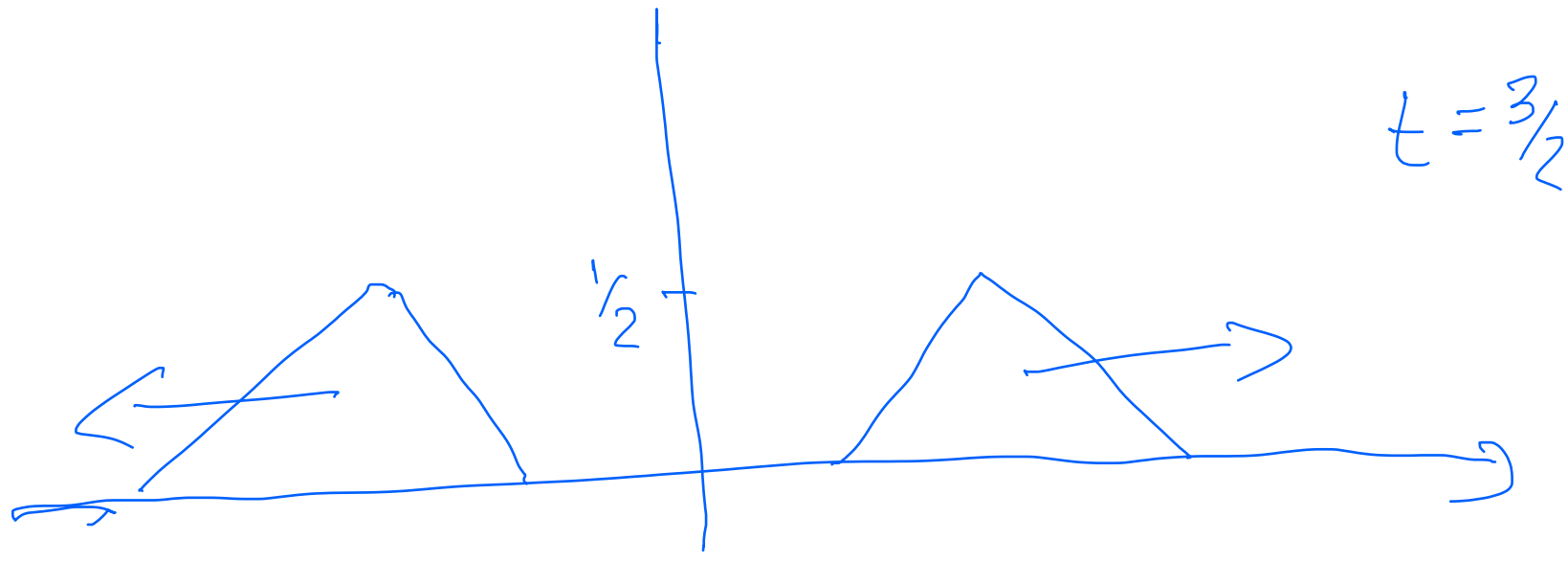
$t = 0$



$t = 1/2$



$t = 1$

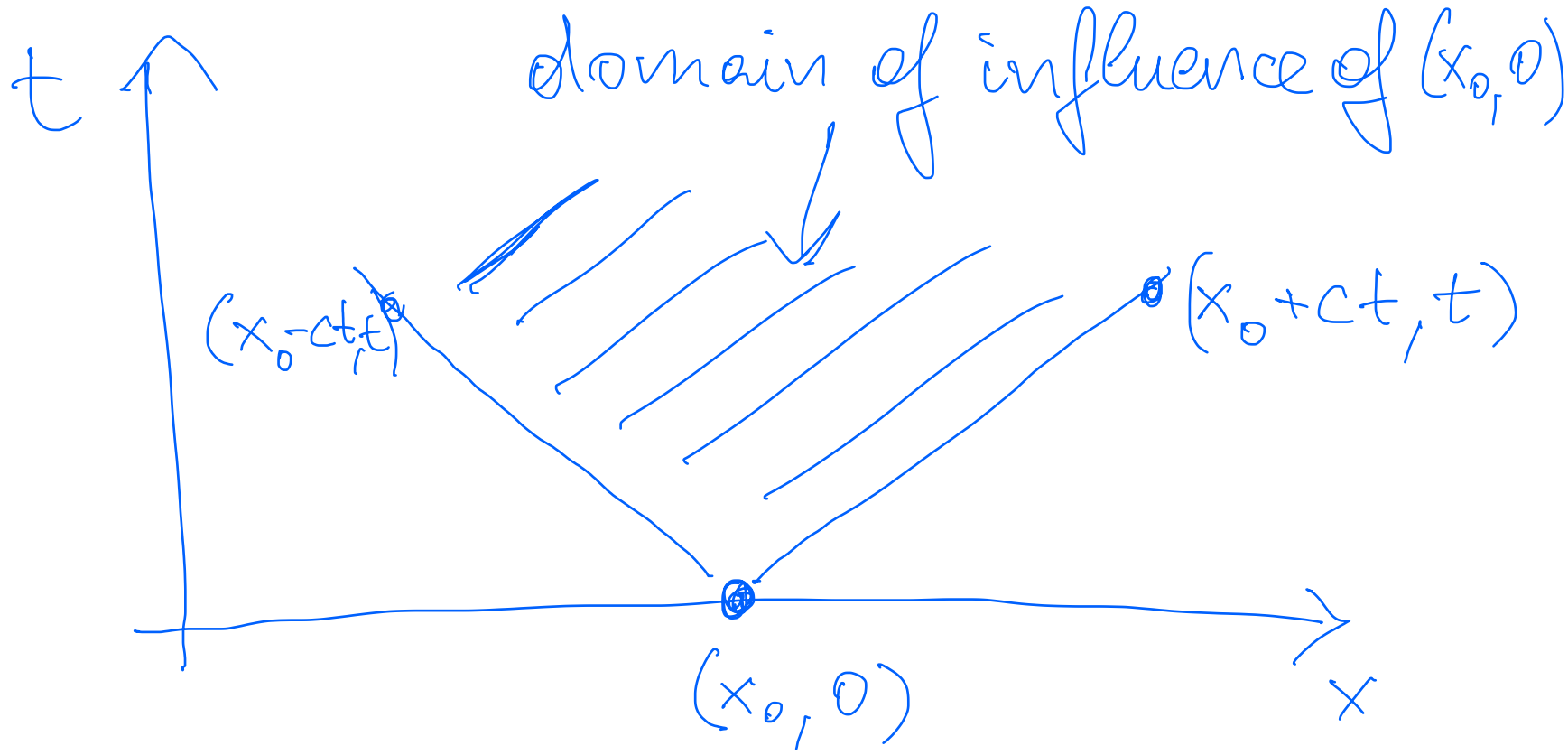


Causality & Energy

recall: the solution of $u_{tt} = c^2 u_{xx}$

with $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$

is $$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$



<u>Exer</u> Suppose ϕ and ψ vanish outside the interval $[-R, R]$. Show that at time t we have $u(x,t) = 0$ for x outside of the interval $[-R-ct, R+ct]$

Assume $|x| > R + ct$

$\Rightarrow |x \pm ct| > R$ (by triangle ineq.)

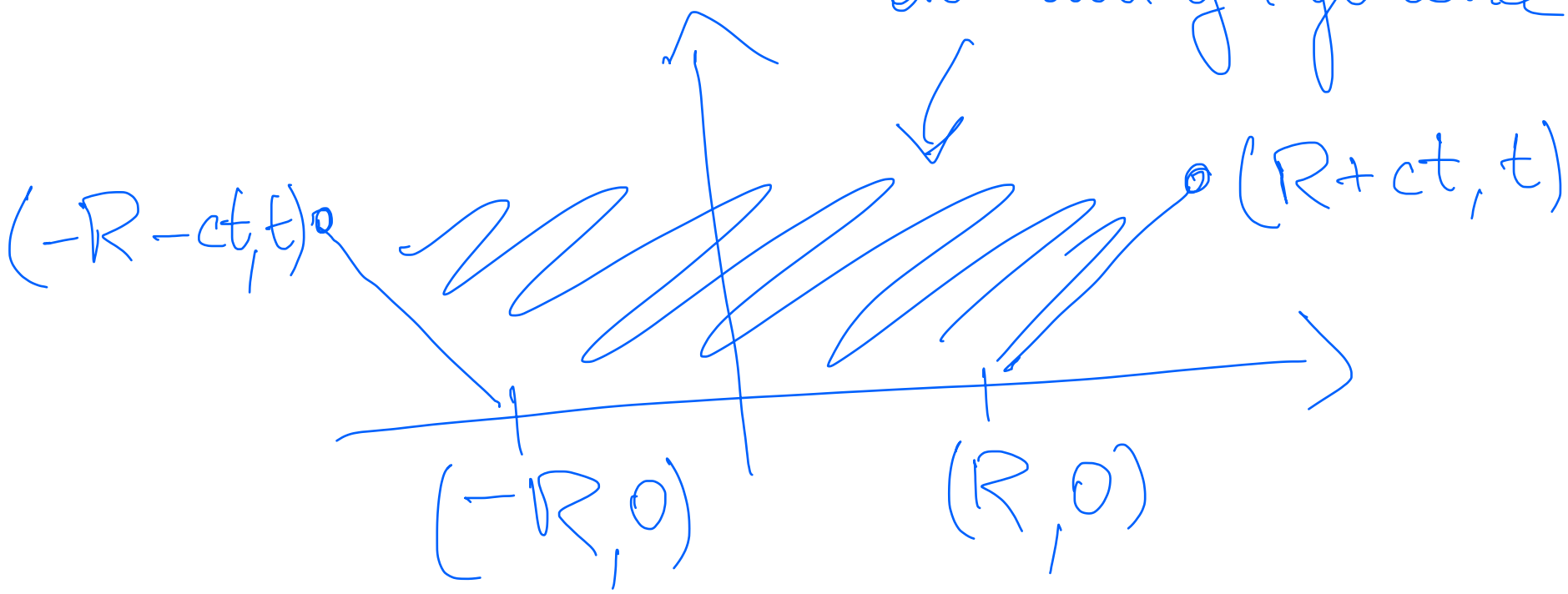
$\Rightarrow \phi(x \pm ct) = 0, \ \psi(s) = 0 \ \forall s \in [x-ct, x+ct]$

assumptions on ϕ, ψ

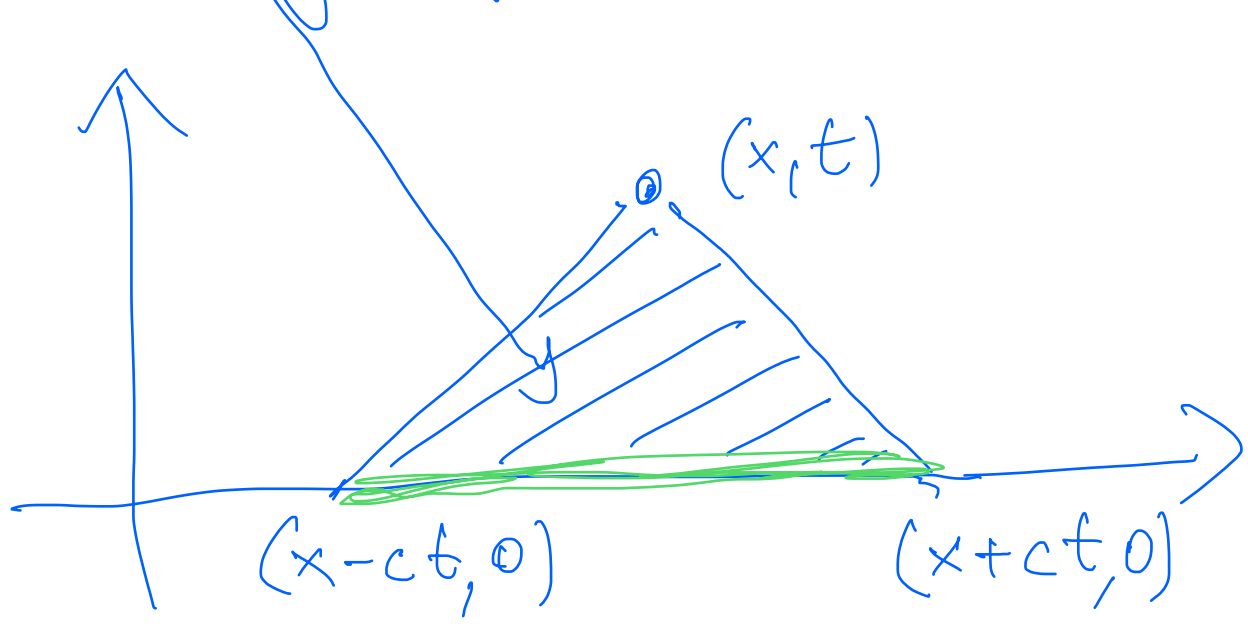
$\Longrightarrow u(x,t) = 0$

sol. formula

domain of influence



domain of dependence / past history

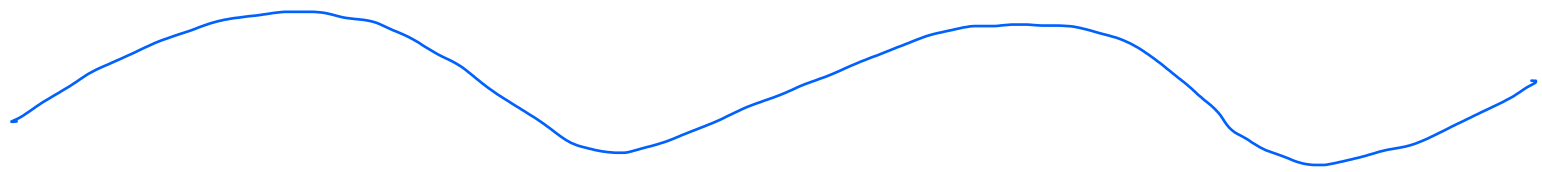


Upshot: Nothing goes faster than speed c.

Energy

(infinite) string with density ρ, tension T.

recall: $u_{tt} = c^2 u_{xx}$ where $c = \sqrt{T/\rho}$



kinetic energy of point mass $= \frac{1}{2} m v^2$

kinetic energy of string $E_{kin} = \frac{1}{2} \int \rho u_t^2 \, dx$

$$\Rightarrow \frac{d}{dt} E_{kin} = \int \rho u_t u_{tt} \, dx \qquad u_{tt} = \frac{T}{\rho} u_{xx}$$

$$= \int T u_t u_{xx} \, dx$$

$$= -T \int u_{tx} u_x \, dx$$

integ. by parts

(assume u vanishes outside interval)

$$= -\frac{T}{2}\int \frac{\partial}{\partial t}\left(u_x^2\right)\,dx$$

$$= -\frac{d}{dt}\underbrace{\frac{1}{2}T\int u_x^2\,dx}_{=:E_{pot}}$$

$$\Rightarrow \boxed{E = \frac{1}{2}\int_{-\infty}^{\infty}\left(\rho u_t^2 + T u_x^2\right)dx \text{ is preserved,}}$$

$$\text{i.e. } \frac{dE}{dt} = 0 \quad \text{(conservation of energy)}$$

Waves in 3-dim space

$$u_{tt} = c^2\left(u_{xx} + u_{yy} + u_{zz}\right)$$

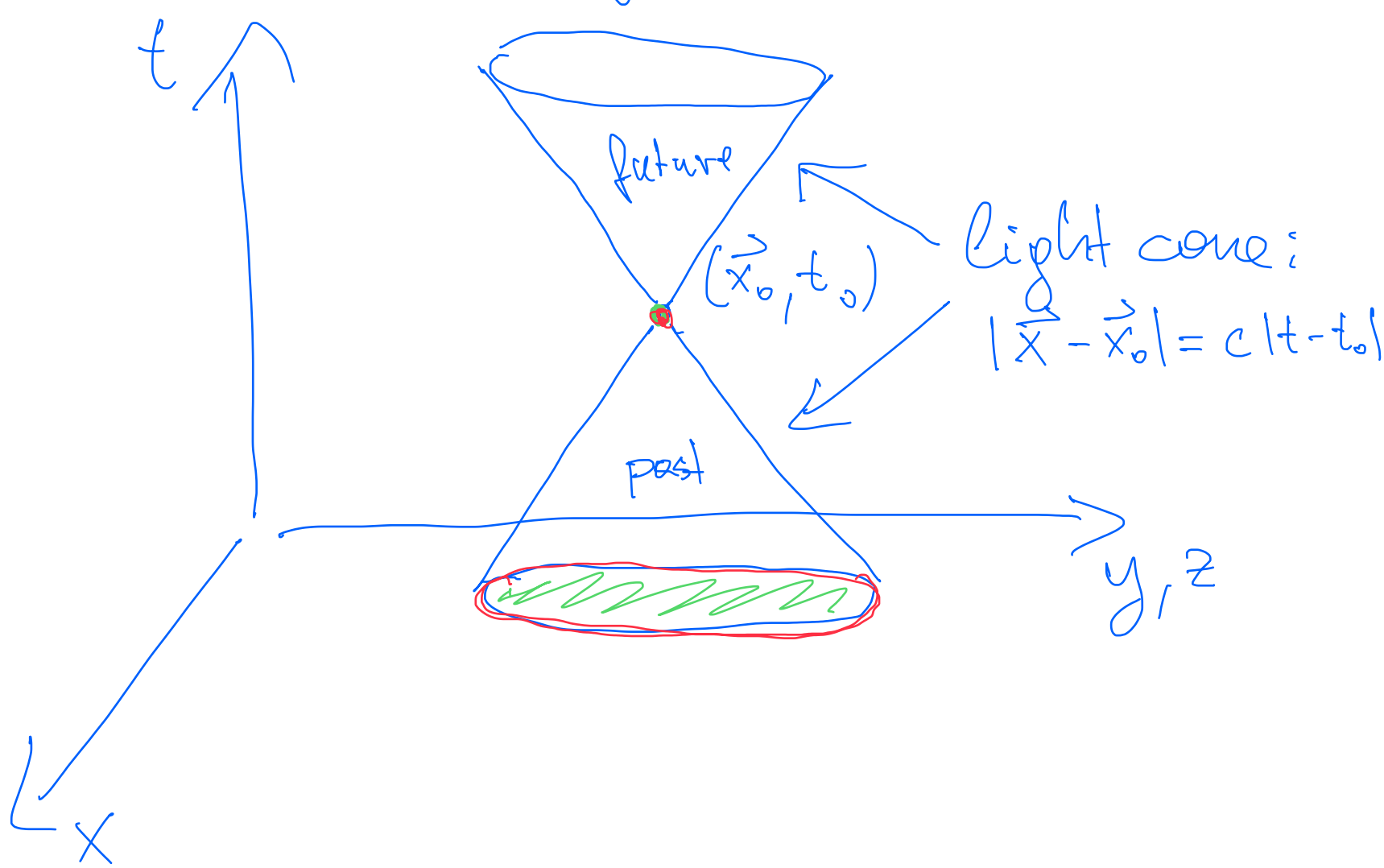
Exer Find a formula for the total energy and show it is preserved.

$$E = \frac{1}{2}\iiint \left(u_t^2 + c^2|\nabla u|^2\right) dx\,dy\,dz$$

Indeed: $$\frac{dE}{dt} = \iiint u_t \underbrace{u_{tt}}_{=c^2\Delta u} + c^2\langle \nabla u, \nabla u_t\rangle$$

$$\underset{IBP}{=} \iiint -c^2\langle \nabla u_t, \nabla u\rangle + c^2\langle \nabla u, \nabla u_t\rangle = 0.$$

<u>Exer</u> Principle of causality in 3d?



<u>Rmk</u> Interesting differences between 2d & 3d (HW)