

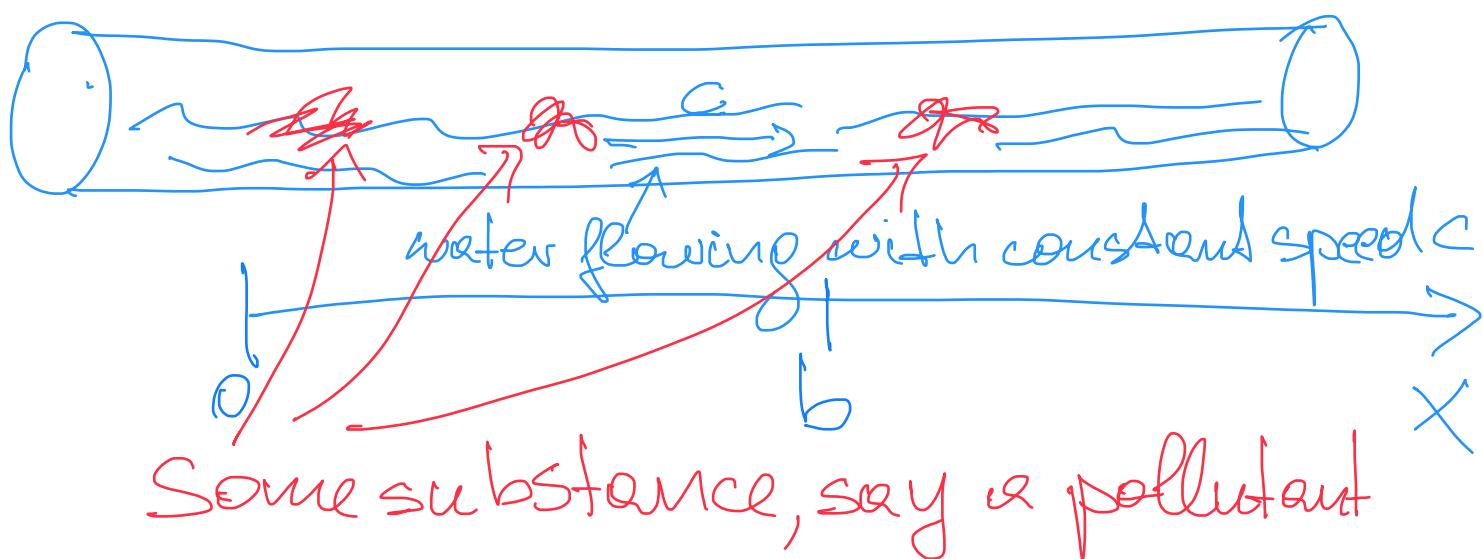
# Where do PDEs come from?

Ex) Transport eqn

$$u_t + cu_x = 0$$

QS model for : traffic flow  
air pollution  
dye dispersion

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$u(x, t)$  = its concentration at position  $x$  at time  $t$ .

Derivation of the eqn:

amount of pollutant in interval  $[0, b]$   
at time  $t$  is  $M = \int_0^b u(x, t) dx$

At time  $t+h$  the particles have moved to the right by  $ch$ , hence

$$M = \int_{ch}^{b+ch} u(x, t+h) dx$$

$$\frac{d}{dt} \Rightarrow u(b, t) = u(b+ch, t+h)$$

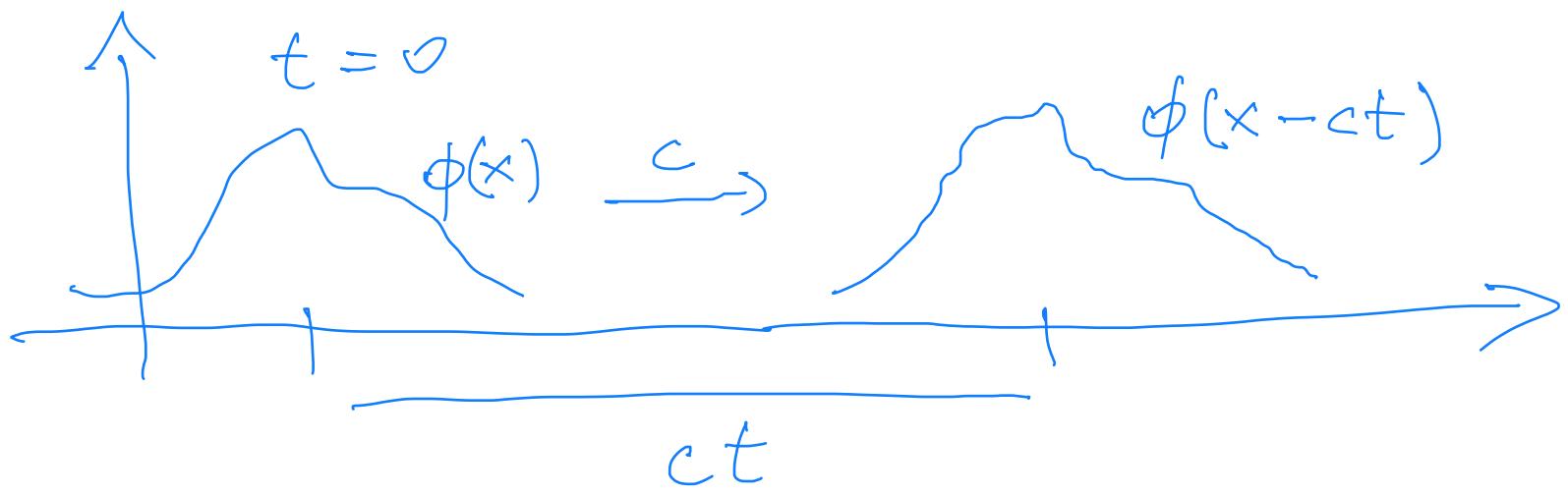
$$(FTOC: \frac{d}{dy} \int_a^y f(x) dx = f(y))$$

$$\left. \frac{d}{dh} \right|_{h=0} \Rightarrow 0 = cu_x(b, t) + u_t(b, t)$$

Exer Solve

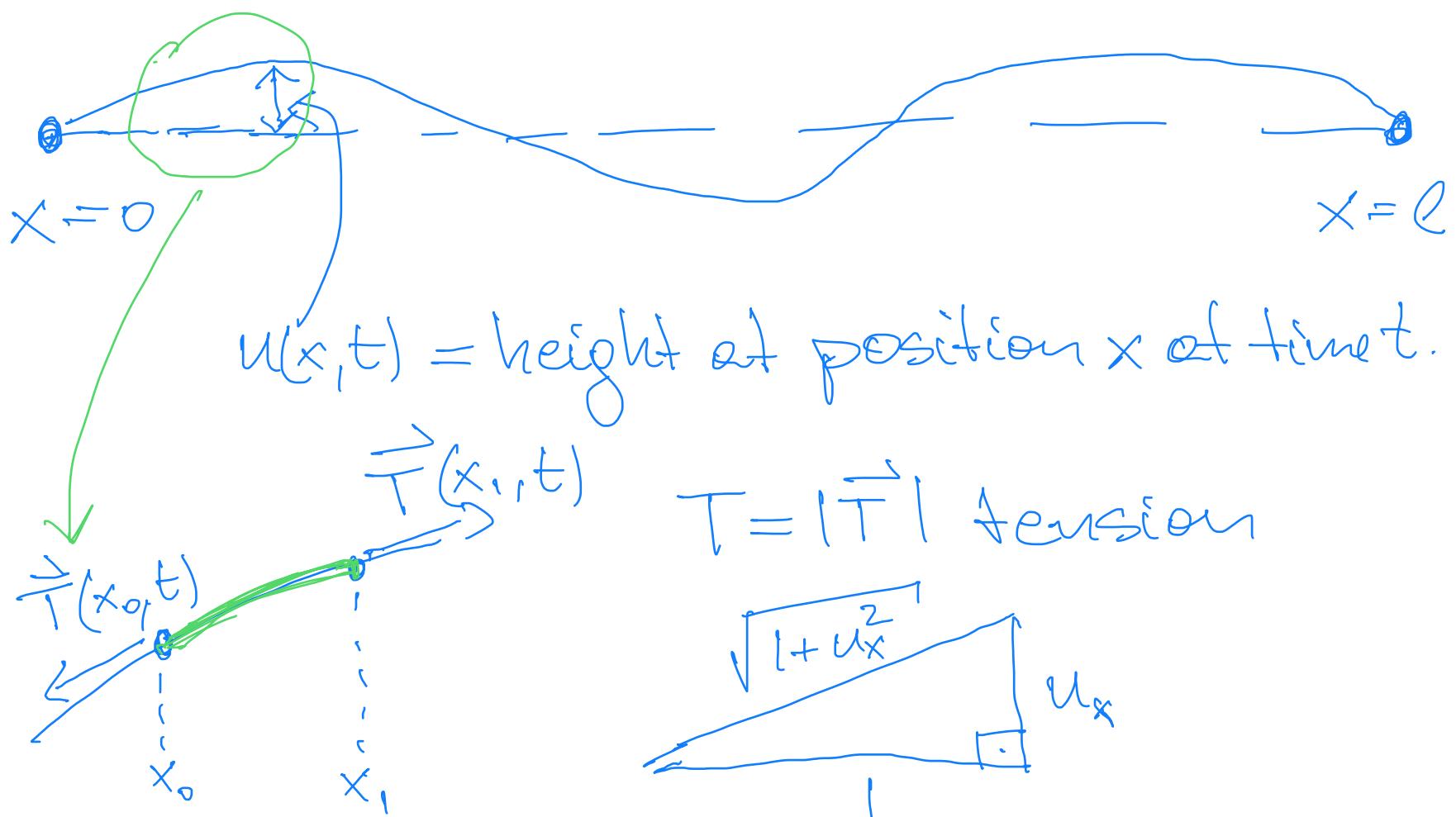
$$\begin{cases} u_t(x, t) + cu_x(x, t) = 0 \\ u(x, 0) = \phi(x) \end{cases}$$

$$\text{Solution: } u(x, t) = \phi(x - ct)$$



## Ex2: Vibrating String

(e.g. guitar, violin)



Newton's law  $\vec{F} = m\vec{a}$  for part of string between  $x_0$  &  $x_1$

longitudinal:  $\frac{T}{\sqrt{1+u_x^2}} \Big|_{x_0}^{x_1} = 0$

transverse:  $\frac{\rho u_x}{\sqrt{1+u_x^2}} \Big|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt} dx$

For  $|u_x| \ll 1$ , then  $\sqrt{1+u_x^2} \approx 1$ , hence

$$T = \text{const}$$

$$\rho u_x \Big|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt} dx$$

$$\Rightarrow \rho u_{xx} = \rho u_{tt}$$

$$\Rightarrow \boxed{u_{tt} = c^2 u_{xx}} \quad c = \sqrt{T/\rho}$$

wave eqn

Puzzle Guess eqn for vibrating drumhead

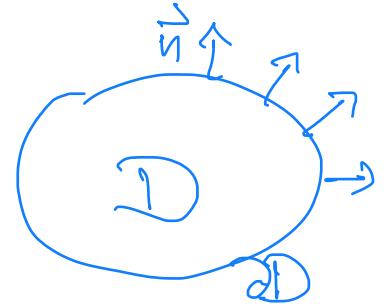


Answer

$$u_{tt} = c^2(u_{xx} + u_{yy})$$

$u(x, y, t)$  = vertical displacement

Newton's law for D



$$\int\limits_{\partial D} T \frac{\partial u}{\partial n} ds = \iint_D \rho u_{tt} dx dy$$

$$\frac{\partial u}{\partial n} = \vec{n} \cdot \nabla u \quad \text{normal derivative of } u$$

Green's theorem

$$\int\limits_{\partial D} \frac{\partial u}{\partial n} ds = \iint_D \operatorname{div}(\nabla u) dx dy$$

$\underbrace{\operatorname{div}(\nabla u)}$

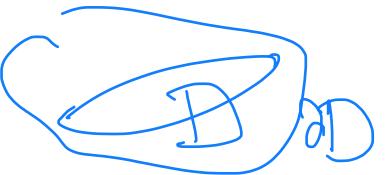
$$= u_{xx} + u_{yy}$$

$$\Rightarrow \iint_D (u_{xx} + u_{yy}) dx dy = \iint_D \frac{\rho}{T} u_{tt} dx dy$$

$$\Rightarrow u_{tt} = c^2(u_{xx} + u_{yy}), \quad c = \sqrt{T/\rho}$$

## Ex5: Heat flow

$u(x, y, z, t)$  the temperature

region  $D$ : 

amount of heat (say in calories)  
contained in  $D$ :

$$H(t) = \iiint_D c_p u \, dx \, dy \, dz$$

$\uparrow \uparrow \uparrow$   
density  
capacity

Say  $c_p = 1$   
for ease  
of notation

Fourier's Law:

heat flow  $\sim$  temperature gradient

$$\Rightarrow \frac{dH}{dt} = \iint_{\partial D} \vec{n} \cdot \nabla u \, dS$$

$$\operatorname{div}(\nabla u) = u_{xx} + u_{yy} + u_{zz}$$

$$\Rightarrow \iiint_D u_t \, dx \, dy \, dz = \iiint_D (u_{xx} + u_{yy} + u_{zz}) \, dx \, dy \, dz$$

$\nabla^2 D$

$$u_t = u_{xx} + u_{yy} + u_{zz}$$

(heat/diffusion eqn)

Ex 6

$$u_{xx} + u_{yy} + u_{zz} = 0$$

(Laplace eqn)

e.g. describes stationary states  
for heat/wave eqn.

Q: Where have you seen the

$$\text{eqn } u_{xx} + u_{yy} = 0 \text{ ?}$$

- physics (electrostatics)
- multivariable calculus
- complex analysis :

real & complex part of any holomorphic function are harmonic (consequence of Cauchy-Riemann eqns).

## Ex 7 Schrödinger eqn

$$\boxed{iu_t = -\Delta u + Vu}$$

$V$  given potential, eg  $V = \frac{1}{r}$  for electron.

$u = u(x, y, z, t) \in \mathbb{C}$  wave function.

$$\iiint_D |u|^2 dx dy dz \quad \begin{matrix} \text{probability to} \\ \text{find electron in} \\ \text{region } D. \end{matrix}$$

and many others

(eg geometric PDEs)

# Initial & boundary conditions

## Initial conditions:

e.g. for heat eqn  $u_t = u_{xx} + u_{yy} + u_{zz}$

$$u(x, y, z, t_0) = \phi(x, y, z)$$

given function: temperature at  
( $x, y, z$ ) at time  $t_0$ .

Exer for wave eqn;

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}$$

initial conditions?

$$\left\{ \begin{array}{l} u(x, y, z, t_0) = \phi(x, y, z) \text{ initial position} \\ u_t(x, y, z, t_0) = \psi(x, y, z) \text{ initial velocity} \end{array} \right.$$

## Boundary conditions:

two most important kinds

Dirichlet condition:  $u$  is specified on  $\partial D$

Neumann condition:  $\frac{\partial u}{\partial n}$  is specified on  $\partial D$

Ex Heat diffusion in a container  $D$



$$u_t = u_{xx} + u_{yy} + u_{zz}$$

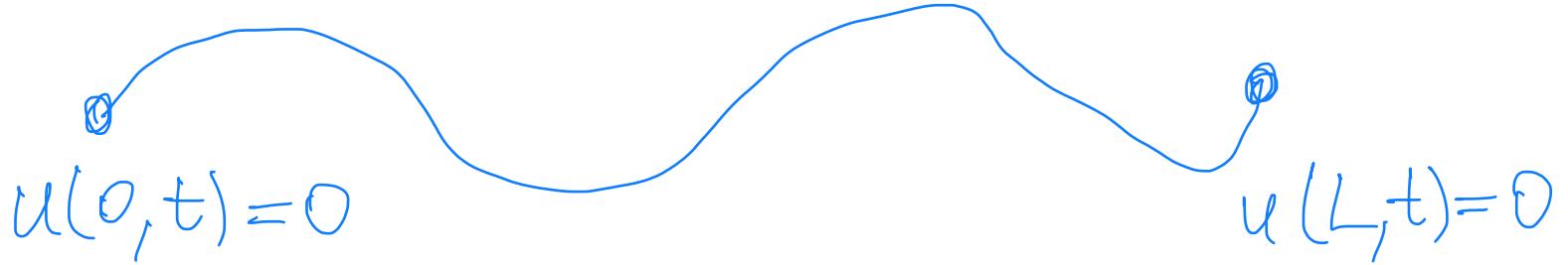
i) perfectly insulated:  $\frac{\partial u}{\partial n} = 0$

.) Immersed in large reservoir of  
specified temperature  $g(x, y, z, t)$   
perfect thermal conductance:

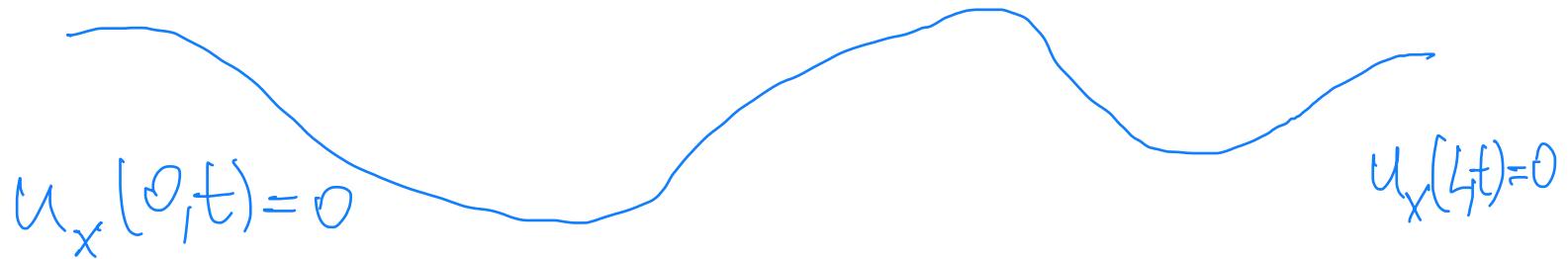
$$u(x, y, z, t) = g(x, y, z, t) \quad \forall (x, y, z) \in \partial D.$$

Exer boundary conditions  
for vibrating string?

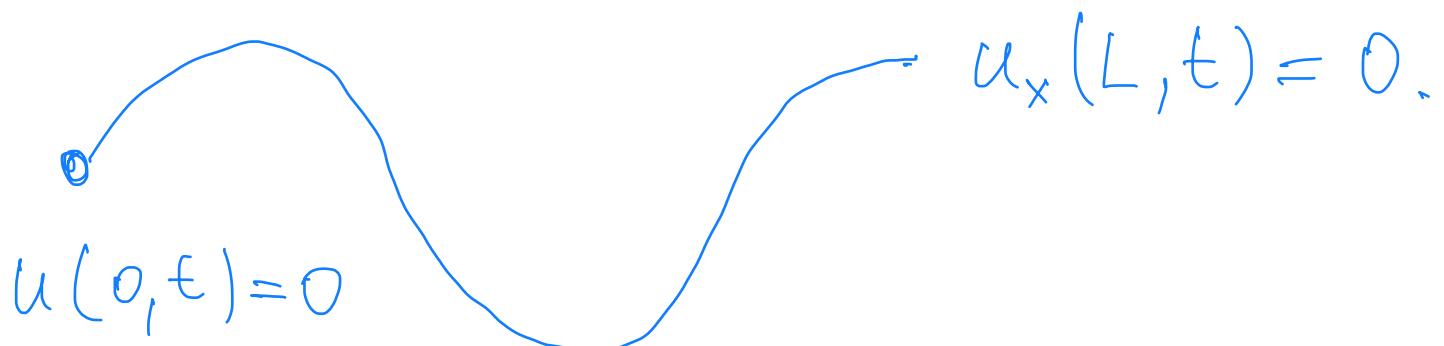
Dirichlet (fixed endpoints)



Neumann (loose endpoints)



Mixed



## Well-posed problems

PDE in a domain together  
with some initial and/or  
boundary condition st:

- (i) Existence: There is at least one solution
- (ii) Uniqueness: there is at most one solution
- (iii) Stability: The unique solution  $u$   
depends in a stable way on the  
data of the problem.

Example The problem



$$\left\{ \begin{array}{l} u_{xx} + u_{yy} = 0 \text{ in } D = \{(x,y) | x^2 + y^2 \leq 1\} \\ u(\vec{x}) = \phi(\vec{x}) \text{ for } \vec{x} \in \partial D \end{array} \right.$$

is well-posed (see later).

Exer Is the problem

$$\left\{ \begin{array}{l} u_{xx} + u_{yy} = 0 \quad \text{in } D \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D \end{array} \right.$$

Well-posed?

No! Uniqueness fails,

e.g.  $u \equiv 1$  &  $u \equiv 2$  are

two different solutions!

Exer Well-posed?

(a)  $\left\{ \begin{array}{l} u_t = u_{xx} \quad x \in \mathbb{R}, t \geq 0 \\ u(x, 0) = \phi(x) \quad x \in \mathbb{R} \\ u_t(x, 0) = \psi(x) \quad x \in \mathbb{R} \end{array} \right.$  No!

(b)  $\left\{ \begin{array}{l} u_t = -u_{xx} \quad x \in \mathbb{R}, t \geq 0 \\ u(x, 0) = \phi(x) \quad x \in \mathbb{R} \end{array} \right.$  No!