

What is a PDE?

Ex $u_t = u_{xx}$

Want to find a function $u = u(x, t)$ such that its time derivative u_t equals its 2nd spatial derivative u_{xx}

Claim $u(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$ ($t > 0$)
is a solution.

Indeed

$$u_t = \frac{1}{\sqrt{4\pi}} \left(\frac{-1/2}{t^{3/2}} \right) e^{-x^2/4t} + \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \left(\frac{x^2}{4t^2} \right)$$

$$= \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \left(\frac{x^2}{4t^2} - \frac{1}{2t} \right)$$

$$u_x = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \cdot \left(-\frac{x}{2t}\right)$$

$$u_{xx} = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \left(\frac{x^2}{4t^2} - \frac{1}{2t}\right) = u_t \quad \square$$

In general, want to find a function depending on more than one variable, that solves some equation depending on the partial derivatives.

e.g. $u = u(x, t)$ or $u = u(x, y)$
or $u = u(x, y, t)$

Ex

(1) $u_t = u_{xx}$ (heat eqn)

(2) $u_{xx} + u_{yy} = 0$ (Laplace's eqn)

$$(3) \quad u_{tt} = u_{xx} + u^3 \quad (\text{nonlinear wave eqn})$$

$$(4) \quad (1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0$$

(minimal surface eqn)

Observation (1) & (2) are linear PDEs
(3) & (4) are nonlinear PDEs

Indeed (1) can be written as

$$\mathcal{L}(u) = 0, \quad \text{where } \mathcal{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2},$$

and it holds that

$$\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v), \quad \mathcal{L}(cu) = c\mathcal{L}(u)$$

Note • $\mathcal{L}(u) = 0, \mathcal{L}(v) = 0 \Rightarrow \mathcal{L}(u+v) = 0$

• $\mathcal{L}(u) = 0, c = \text{const} \Rightarrow \mathcal{L}(cu) = 0.$

Above we rewrote our PDE
as follows:

$$u_t = u_{xx}$$

$$\Leftrightarrow u_t - u_{xx} = 0$$

$$\Leftrightarrow \frac{\partial}{\partial t} u - \frac{\partial^2}{\partial x^2} u = 0$$

$$\Leftrightarrow \underbrace{\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right)}_{=: \mathcal{L}} u = 0$$

Exer Linear or nonlinear?

a) $u_{xx} + \sqrt{1+u_x^2} u_{yy} = 0$

b) $u_t = u_{xx} + x^2 \cdot u$

Indeed b) is linear:

$$\left. \begin{aligned} u_t &= u_{xx} + x^2 u \\ v_t &= v_{xx} + x^2 v \end{aligned} \right] +$$

$$\Rightarrow (u+v)_t = (u+v)_{xx} + x^2 (u+v)$$

and similarly

$$\left. \begin{aligned} u_t &= u_{xx} + x^2 u \\ c &= \text{const} \end{aligned} \right\} \Rightarrow (cu)_t = (cu)_{xx} + x^2 (cu)$$

\square

Exer Find all $u(x, y)$ solving $u_{xx} = 0$.

$$u_{xx}(x, y) = 0$$

$$\Rightarrow u_x(x, y) = f(y) \quad \text{for some } f$$

$$\Rightarrow \underline{u(x, y) = f(y)x + g(y)}$$

for some f and g .

First order linear PDEs

first order; only first derivatives

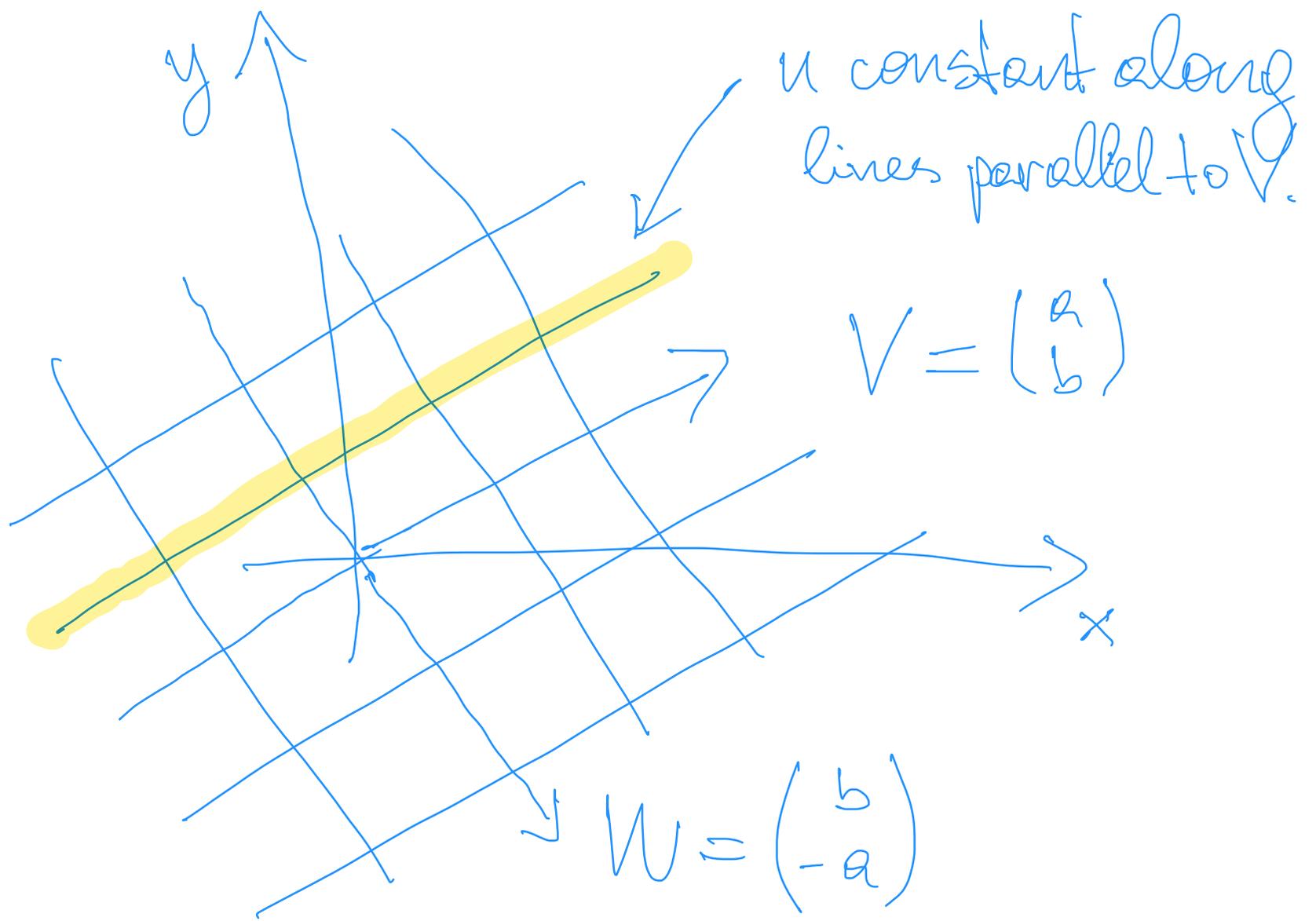
The constant coefficient eqn

$$a u_x + b u_y = 0$$

Geometric approach:

$a u_x + b u_y$ is the directional derivative of u in direction of the vector $V = \begin{pmatrix} a \\ b \end{pmatrix}$

$a u_x + b u_y = 0 \Rightarrow u(x, y)$ does not change in direction V .



lines parallel to V satisfy $bx - ay = c$
 solution is constant on each
 such line (characteristic lines)

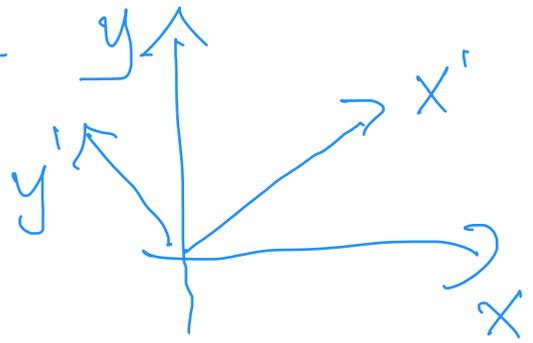
$\Rightarrow u(x, y)$ only depends on $bx - ay$

$\Rightarrow \underline{u(x, y) = f(bx - ay)}$ for some f .

Computational approach:

$$x' = ax + by$$

$$y' = bx - ay$$



Compute (using the chain rule):

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'} \underbrace{\frac{\partial x'}{\partial x}}_{=a} + \frac{\partial u}{\partial y'} \underbrace{\frac{\partial y'}{\partial x}}_{=b} = au_{x'} + bu_y$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial y} = bu_{x'} - au_{y'}$$

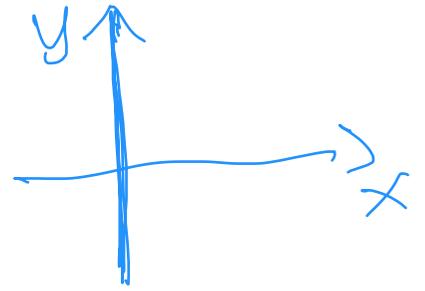
$$\begin{aligned} \Rightarrow au_x + bu_y &= a^2 u_{x'} + abu_{y'} + b^2 u_{x'} - abu_{y'} \\ &= (a^2 + b^2) u_{x'} \end{aligned}$$

$$au_x + bu_y = 0 \Rightarrow u_{x'} = 0$$

assume $a^2 + b^2 \neq 0$

$$\Rightarrow \underline{u = f(y') = f(bx - ay)}$$

Exer Solve $4u_x - 3u_y = 0$ together
with the auxiliary condition $u(0, y) = y^3$



$$4u_x - 3u_y = 0$$

$$\Rightarrow u(x, y) = f(-3x - 4y)$$

$$u(0, y) = f(-4y) = y^3$$

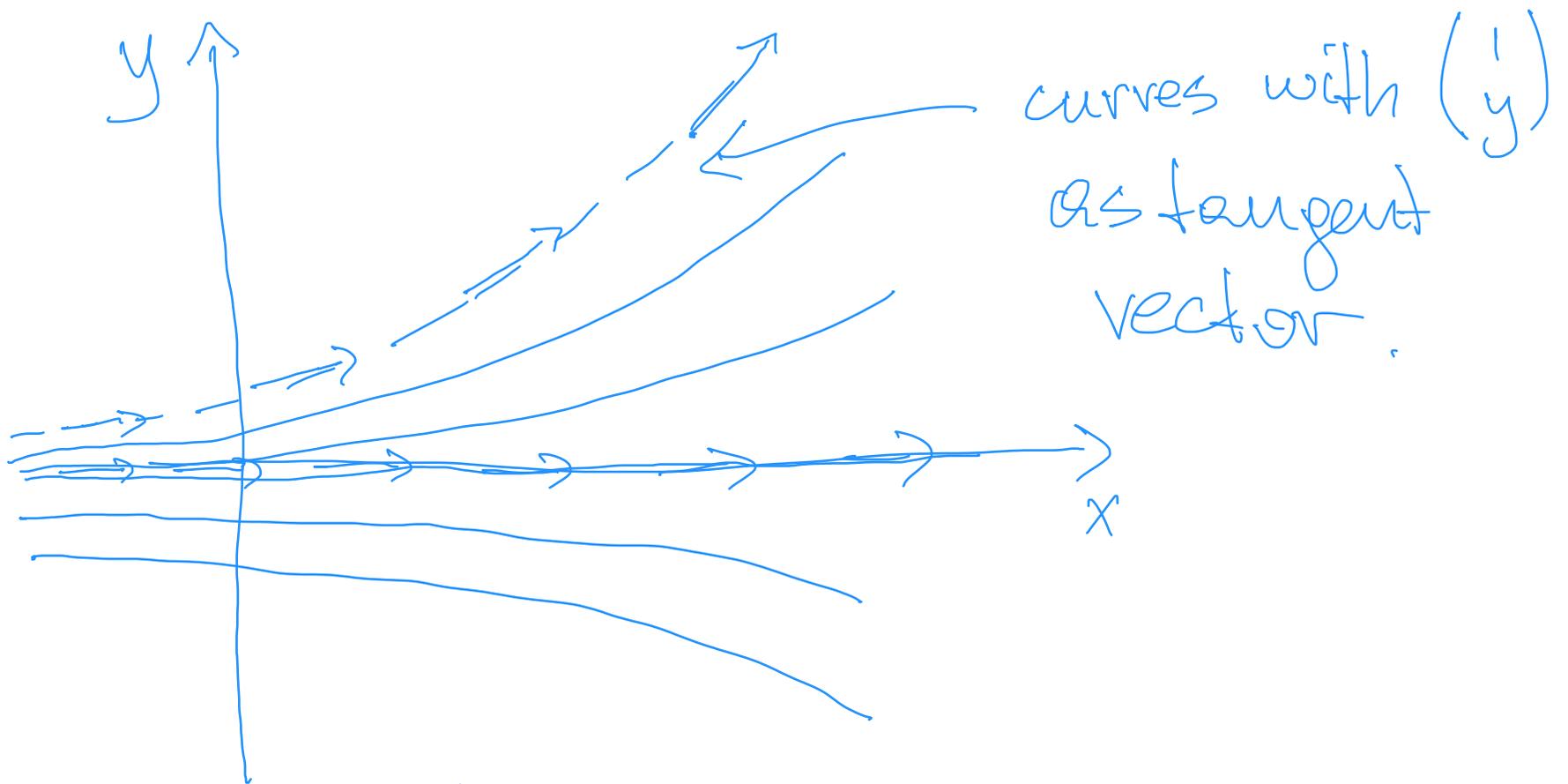
$$w = -4y \Rightarrow f(w) = -\frac{w^3}{64}$$

$$\Rightarrow u(x, y) = (3x + 4y)^3 / 64 .$$

The variable coefficient eqn

Ex $u_x + y u_y = 0$

Geometrically: directional derivative
along $V = \begin{pmatrix} 1 \\ y \end{pmatrix}$ vanishes



compute these curves:

$$\text{slope} = y$$

$$\frac{dy}{dx} = y \Rightarrow \underline{y = C e^x}$$

hence we found the
"characteristic curves" $y = ce^x$.

Observe that:

$$\frac{d}{dx} u(x, ce^x) = u_x + ce^x u_y = u_x + y u_y = 0.$$

$$\Rightarrow u(x, ce^x) = u(0, c)$$

$$\Rightarrow \boxed{u(x, y) = f(e^{-x}y)}$$

$c = e^{-x}y$

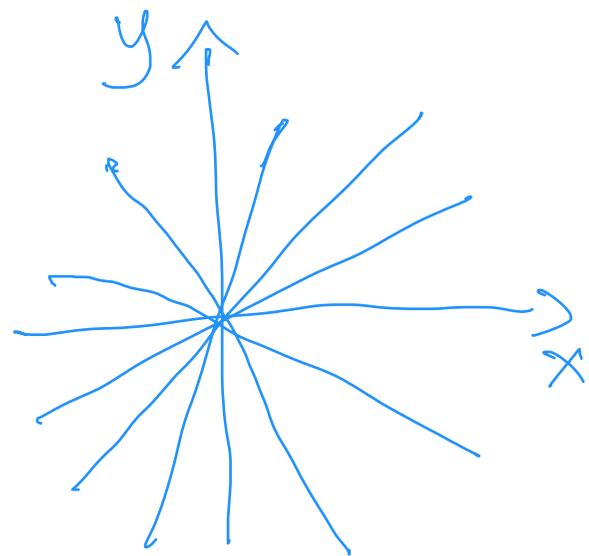
Exer Solve $xu_x + yu_y = 0$

$$\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{=: V} \cdot \begin{pmatrix} u_x \\ u_y \end{pmatrix} = 0$$

char. curves: $\frac{dy}{dx} = \frac{y}{x} \Rightarrow y = cx$

$$\Rightarrow u(x, cx) = u(0, 0) = \text{const}$$

$$\Rightarrow \underline{u(x, y) = \text{const}}$$



Exer Solve $\sqrt{1-x^2} u_x + u_y = 0$

with the condition $u(0, y) = y$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow y = \arcsin x + c$$

$$\Rightarrow u(x, \arcsin x + c) = u(0, c) =: f(c)$$

$$\Rightarrow \underline{u(x, y) = f(y - \arcsin x)}$$