

# Boundary value problems

physically realistic case of finite interval  
(instead of the whole real line)

## Ex : Dirichlet problem for wave eqn

$$\left\{ \begin{array}{l} u_{tt} = c^2 u_{xx} \quad \text{for } 0 < x < l \\ u(0, t) = 0 = u(l, t) \\ u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x) \end{array} \right.$$

## Separation of variables

Look for solutions of the form

$$u(x, t) = X(x)T(t).$$

plug in PDE  $\Rightarrow X(x)T''(t) = c^2 X''(x)T(t)$

divide by  $-c^2 X T$

$$\Rightarrow -\frac{T''}{c^2 T} = -\frac{X''}{X} = : \lambda$$

independent of  $x$

independent of  $t$

$\Rightarrow \lambda$  must be a constant!

Moreover,  $\lambda > 0$  (see later).

Write  $\lambda = \beta^2$ , where  $\beta > 0$ . Get:

$$X'' + \beta^2 X = 0, \quad T'' + c^2 \beta^2 T = 0.$$

These ODEs are easy to solve:

$$X(x) = C \cos(\beta x) + D \sin(\beta x)$$

$$T(t) = A \cos(\beta c t) + B \sin(\beta c t).$$

---

$$u(x, t) = X(x)T(t).$$

Boundary conditions  $u(0, t) = 0 = u(l, t)$

$$\Rightarrow 0 = X(0) = C, 0 = X(l) = D \sin(\beta l)$$

For  $D \neq 0$ , we must have

$$\sin(\beta l) = 0 \Rightarrow \beta l = n\pi,$$

i.e.  $X_n(x) = \sin\left(\frac{n\pi x}{l}\right) \quad (n=1, 2, 3, \dots)$

are solutions, where

$$\lambda_n = \beta_n^2 = \left(\frac{n\pi}{l}\right)^2$$

Thus, get:

$$u_n(x, t) = \left( A_n \cos\left(\frac{n\pi c t}{e}\right) + B_n \sin\left(\frac{n\pi c t}{e}\right) \right) \cdot \sin\left(\frac{n\pi x}{e}\right)$$

is a solution for each  $n = 1, 2, 3, \dots$

Equation is linear  $\Rightarrow$

$$u(x, t) = \sum_n \left( A_n \cos\left(\frac{n\pi c t}{e}\right) + B_n \sin\left(\frac{n\pi c t}{e}\right) \right) \cdot \sin\left(\frac{n\pi x}{e}\right)$$

is a solution.

Initial conditions?

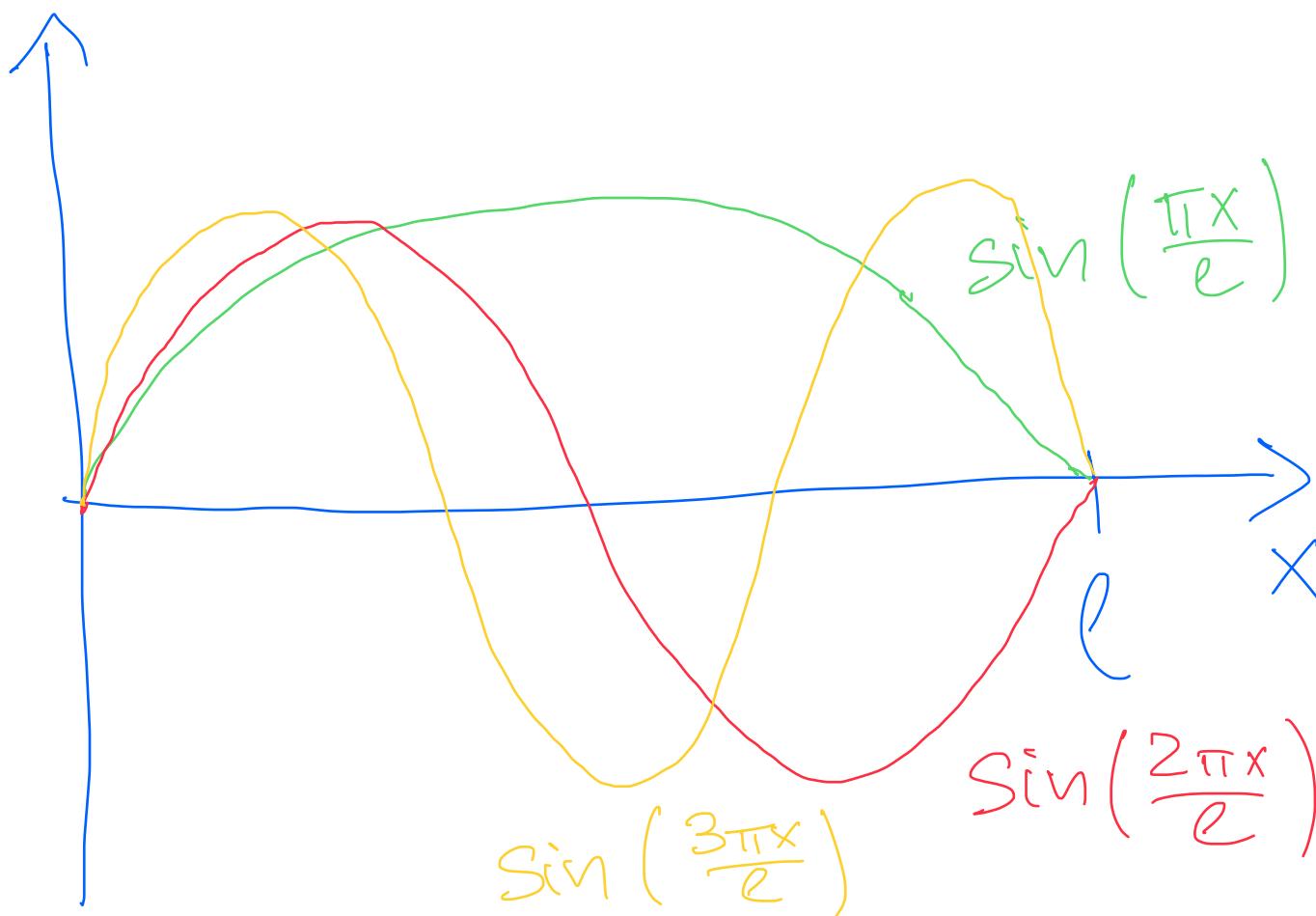
$$\phi(x) = \sum_n A_n \sin\left(\frac{n\pi x}{e}\right)$$

$$\psi(x) = \sum_n B_n \frac{n\pi c}{e} \sin\left(\frac{n\pi x}{e}\right)$$

Usually not possible for finite sums,  
to find  $A_n, B_n$  such that this holds,

but possible for infinite sums,

cf Fourier analysis (next lecture)



Any continuous function  $\phi(x)$  with  $\phi(0)=0=\phi(l)$   
can be expressed as

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{e}\right)$$

for some unique coefficients  $A_n$ .

Ex: vibrating string



$c = \sqrt{T/\rho}$ . Hence  $\frac{n\pi c}{l}$  is given

by the frequencies  $\frac{n\pi \sqrt{T/\rho}}{l}$  ( $n=1, 2, 3, \dots$ )

"fundamental tone"  $\frac{\pi \sqrt{T/\rho}}{l}$

"overtones"  $\frac{2\pi \sqrt{T/\rho}}{l}, \frac{3\pi \sqrt{T/\rho}}{l}, \dots$

exactly double, triple, ... of  
fundamental tone.

Every sound can be expressed as  
combination of these tones.

Exer: Dirichlet problem for heat eqn

$$\left. \begin{array}{l} u_t = u_{xx} \quad (0 < x < l, 0 < t < \infty) \\ u(0, t) = 0 = u(l, t) \\ u(x, 0) = \phi(x) \end{array} \right\}$$

Solve this using separation  
of variables.

$$u(x, t) = X(x)T(t)$$

$$\Rightarrow XT' = X''T$$

$$\Rightarrow \frac{T'}{T} = \frac{X''}{X} = -\lambda = \text{const}$$
$$(\lambda > 0)$$

$$\Rightarrow \begin{cases} X_n(x) = \sin\left(\frac{n\pi x}{e}\right), \quad \lambda_n = \left(\frac{n\pi}{e}\right)^2 \\ T_n(t) = e^{-\left(\frac{n\pi}{e}\right)^2 t} \end{cases}$$

$$\Rightarrow u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{e}\right)^2 t} \sin\left(\frac{n\pi x}{e}\right)$$

where  $A_n$  determined

$$\text{by } \phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{e}\right).$$

$$X'' = -\lambda X$$

Q: Is it always the case that  $\lambda > 0$ ?

$$\text{Try } \lambda = 0: \quad X'' = 0$$

$$\Rightarrow X(x) = C + Dx$$

$$X(0) = 0 = C,$$

$$X(\ell) = 0 = D\ell \Rightarrow D = 0. \\ \Rightarrow X \equiv 0.$$

Try  $\lambda < 0$ : Write  $\lambda = -\gamma^2$

$$\Rightarrow X'' = \gamma^2 X$$

$$\Rightarrow X(x) = C \cosh(\gamma x) + D \sinh(\gamma x)$$

$$0 = X(0) = C$$

$$0 = X(\ell) = D \sinh(\gamma \ell) \Rightarrow D = 0.$$

$$\Rightarrow X \equiv 0.$$

Hence,  $\lambda > 0$ .

Another way to think about it

$$\begin{cases} -\frac{d^2}{dx^2} X = \lambda X & \text{"eigenvalue problem"} \\ X(0) = 0 = X(l) \end{cases}$$

all eigenvalues are positive. They

are  $\lambda_n = \left(\frac{n\pi}{l}\right)^2$  ( $n=1, 2, 3, \dots$ )

with eigenfunctions  $X_n(x) = \sin\left(\frac{n\pi x}{l}\right)$

Again why  $\lambda > 0$ :

$$-X'' = \lambda X$$

| multiply by  $X$   
and integrate

$$\Rightarrow \int_0^l -X X'' dx = \int_0^l \lambda X^2 dx = \lambda \int_0^l X^2 dx$$

$$X(0)=0=X(l) \Rightarrow \int_0^l X'^2 dx \Rightarrow \lambda > 0.$$

## Neumann condition

Exer

Solve

$$\left\{ \begin{array}{l} u_t = u_{xx} \quad (0 < x < l, 0 < t < \infty) \\ u_x(0, t) = 0 = u_x(l, t) \\ u(x, 0) = \phi(x) \end{array} \right.$$

$$u(x, t) = X(x)T(t)$$

$$\Rightarrow -X'' = \lambda X, X'(0) = X'(l) = 0, T' = -\lambda T$$

(Boundary conditions)

eigenvalues  $0, \left(\frac{\pi}{l}\right)^2, \left(\frac{2\pi}{l}\right)^2, \dots$

eigenfunctions  $1, \cos\left(\frac{\pi x}{l}\right), \cos\left(\frac{2\pi x}{l}\right), \dots$

$$\Rightarrow u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 t} \cos\left(\frac{n\pi x}{l}\right)$$

where  $\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$ .

Exer Solve

$$\left\{ \begin{array}{l} u_{tt} = c^2 u_{xx} \quad (0 < x < l, 0 < t < \infty) \\ u_x(0, t) = 0 = u_x(l, t) \\ u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x) \end{array} \right.$$

$$u(x, t) = X(x) T(t)$$

$$\Rightarrow \left\{ \begin{array}{l} \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda \\ X'(0) = X'(l) = 0 \end{array} \right.$$

$$\lambda = 0: \quad X(x) = 1, \quad T(t) = A + Bt$$

$$\lambda > 0: \quad X_n(x) = \cos\left(\frac{n\pi x}{l}\right), \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2$$

$$T_n''(t) = -\left(\frac{n\pi c}{l}\right)^2 T_n(t)$$

$$\Rightarrow T_n(t) = A_n \cos\left(\frac{n\pi c}{l} t\right) + B_n \sin\left(\frac{n\pi c}{l} t\right)$$

Thus:

$$u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \left( A_n \cos\left(\frac{n\pi c t}{\ell}\right) + B_n \sin\left(\frac{n\pi c t}{\ell}\right) \right) \cdot \cos\left(\frac{n\pi x}{\ell}\right)$$

where A's & B's are determined by

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right)$$

$$\psi(x) = \frac{B_0}{2} + \sum_{n=1}^{\infty} \frac{n\pi c}{\ell} B_n \cos\left(\frac{n\pi x}{\ell}\right).$$