

The diffusion equation

$$\boxed{u_t = u_{xx}}$$

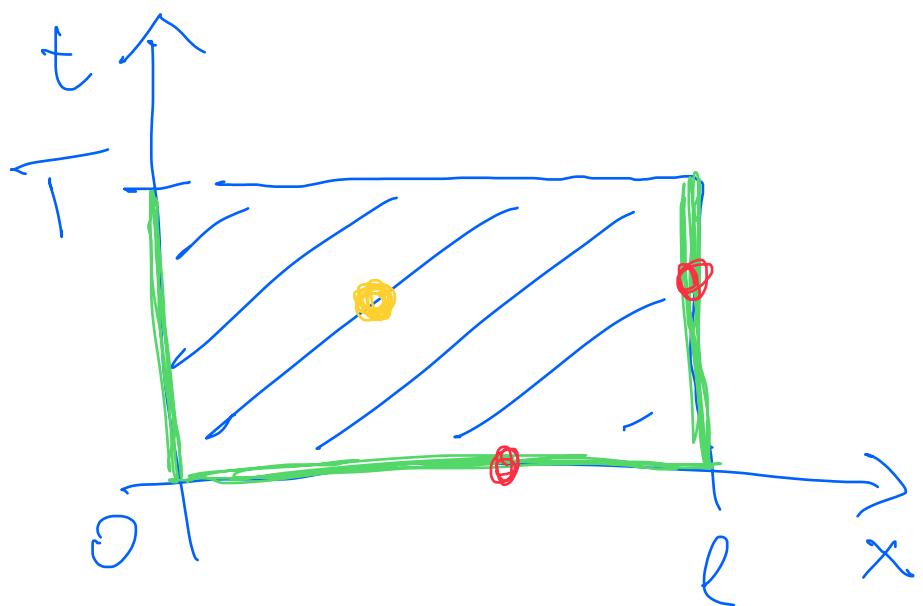
think of $u(x,t)$ as temperature
at position x at time t .

General properties

(formulas for solution later if $x \in \mathbb{R}$)

Maximum principle

If $u(x,t)$ satisfies the diffusion equation in the rectangle $0 \leq x \leq l, 0 \leq t \leq T$, then the maximum value of $u(x,t)$ is attained either initially ($t=0$) or on the lateral sides ($x=0$ or $x=l$).



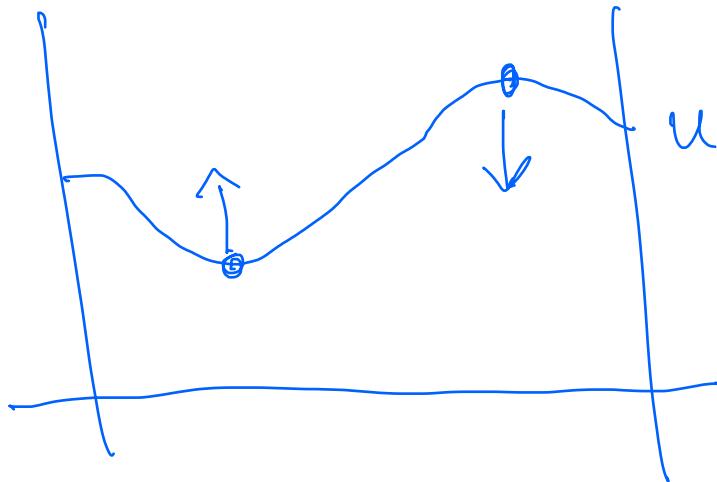
Cor Same holds for minimum.

Proof of Cor Apply max. princ. to $-u(x,t)$.



Physical intuition heat: max ↓, min ↑
in time.

Math intuition

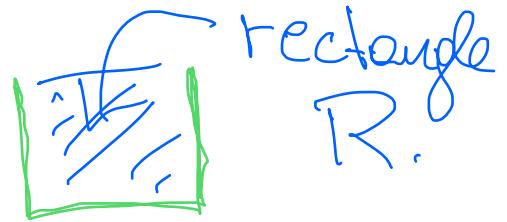


$u_{xx} < 0$ at spatial max

$u(x,t)$ $u_{xx} > 0$ at spatial min

but $u_{xx} = u_t = 0$
at interior min/max.

Proof of the max. princ.



Let $M :=$ maximum value of $u(x,t)$ on the three sides $t=0, x=0, x=\ell$

(exists by continuity & compactness)

want to show: $u(x,t) \leq M \quad \forall (x,t) \in R$

Let $\varepsilon > 0$ and consider

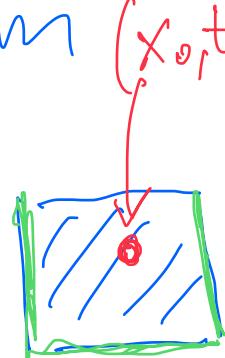
$$v(x,t) := u(x,t) + \varepsilon x^2.$$

Then $v(x,t) \leq M + \varepsilon \ell^2$ on the three sides.

Compute:

$$\underline{v_t - v_{xx}} = \underline{u_t - u_{xx} - 2\varepsilon} = \underline{-2\varepsilon} < 0$$

Suppose towards a contradiction
that $v(x,t)$ attains its maximum (x_0, t_0)
at $(x_0, t_0) \in \text{Int}(R)$.



Calculus $\Rightarrow v_t(x_0, t_0) = 0, v_{xx}(x_0, t_0) \leq 0$

$$\Rightarrow (v_t - v_{xx})|_{(x_0, t_0)} \geq 0 \quad \Downarrow$$

Similarly, if max at (x_0, t_0) with

$$0 < x_0 < l, t_0 = \bar{T},$$



then $v_t(x_0, t_0) \geq 0, v_{xx}(x_0, t_0) \leq 0$

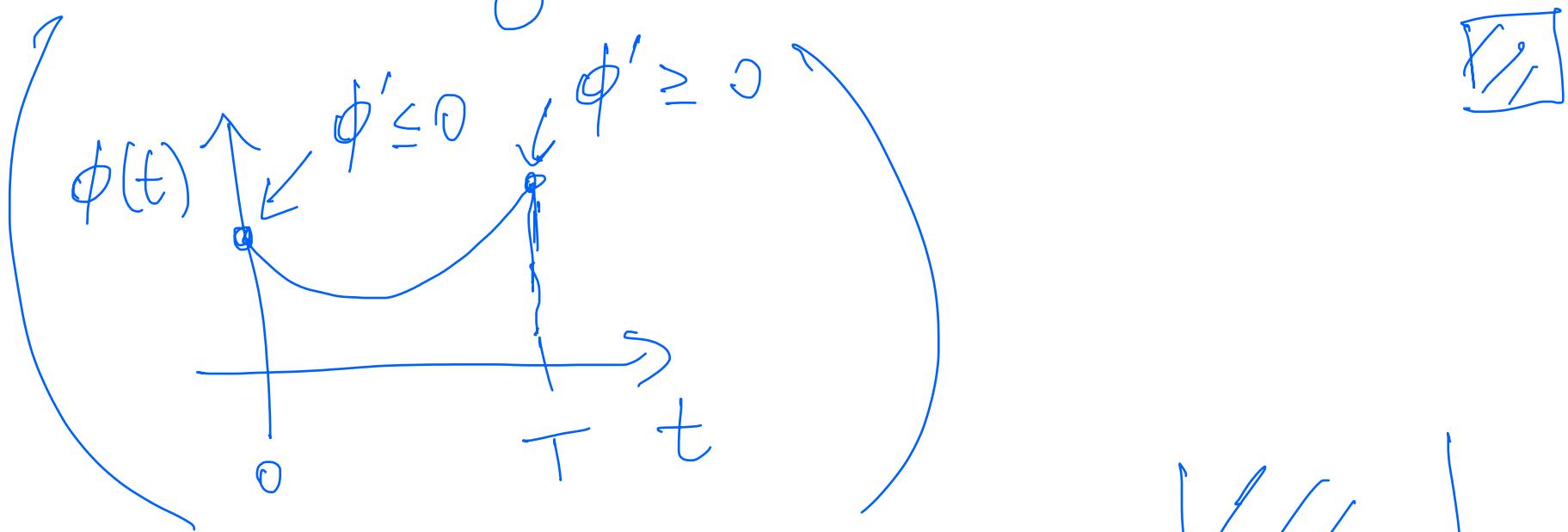
$$\Rightarrow (v_t - v_{xx})|_{(x_0, t_0)} \geq 0 \quad \Downarrow$$

Hence, the max of $v(x,t)$ must be
attained on bottom or lateral sides.

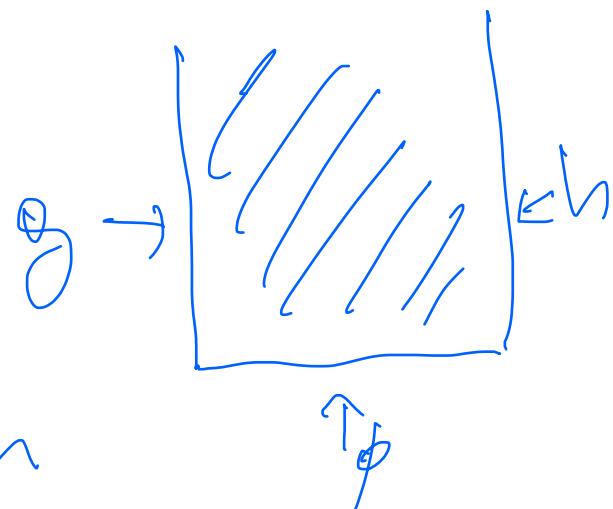
$$\Rightarrow v(x,t) \leq M + \epsilon l^2 \quad \forall (x,t) \in R.$$

$$\Rightarrow u(x,t) + \varepsilon x^2 \leq M + \varepsilon l^2 \quad \forall (x,t) \in \mathbb{R}$$

arbitrary $\Rightarrow u(x,t) \leq M \quad \forall (x,t) \in \mathbb{R}$



Exer (Uniqueness)



Show that the problem

$$\left\{ \begin{array}{l} u_t - u_{xx} = f(x,t) \quad 0 < x < l, t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t), u(l,t) = h(t) \end{array} \right.$$

has at most one solution.

Always start uniqueness proof with:

Suppose u_1, u_2 are two solutions.

Consider $w := u_1 - u_2$.

$$\Rightarrow \left\{ \begin{array}{l} w_t - w_{xx} = 0 \\ w(x, 0) = 0 \\ w(0, t) = 0, \quad w(l, t) = 0 \end{array} \right.$$

$w = 0$ on bottom & sides

Max/min princ. $\Rightarrow w \equiv 0$, i.e. $u_1 \equiv u_2$ \square

Alternative approach (energy method)

$$\frac{d}{dt} \int_0^l \frac{1}{2} w(x, t)^2 dx = \int_0^l w w_t dx = \int_0^l w w_{xx} dx$$

$$= \underbrace{w w_x|_0^l}_{=0} - \int_0^l w_x^2 dx \leq 0.$$

$$\Rightarrow \int_0^l w(x,t)^2 dx \leq \int_0^l w(x,0)^2 dx = 0$$

$$\Rightarrow w \equiv 0, \text{ i.e. } u_1 \equiv u_2$$

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Stability

Consider the problem

$$\left\{ \begin{array}{ll} u_t = u_{xx} & 0 < x < l, t > 0 \\ u(x,0) = \phi(x) & \\ u(0,t) = 0 = u(l,t) & \end{array} \right. .$$

Q: well-posed?

•) existence ✓ (see later)

•) uniqueness ✓

•) stability

u_1 , sol. with init. cond ϕ_1

u_2 — \parallel ϕ_2 .

Max/min principle

$$\Rightarrow \max_{0 \leq x \leq l} |u_1(x, t) - u_2(x, t)| \leq \max_{0 \leq x \leq l} |\phi_1(x) - \phi_2(x)|$$

for all $t > 0$.

("Stability in uniform sense").

Energy method

$$\Rightarrow \int_0^l [u_1(x, t) - u_2(x, t)]^2 dx \leq \int_0^l [\phi_1(x) - \phi_2(x)]^2 dx$$

for all $t > 0$.

("Stability in square integral sense").

Exer Consider the problem

$$\left\{ \begin{array}{l} u_t = u_{xx} \quad 0 < x < 1, t > 0 \\ u(x, 0) = 4x(1-x) \\ u(0, t) = 0 = u(1, t) \end{array} \right.$$

(a) Show $0 < u(x, t) < 1$ for $t > 0, 0 < x < 1$.

(b) Show $u(x, t) = u(1-x, t)$.

(c) Show $t \mapsto \int_0^1 u^2 dx$ is decreasing.

Solution (a) $4x(1-x)$ for $x \in [0, 1]$

has $\min = 0, \max = 1$

min/max princ $\Rightarrow 0 < u(x, t) < 1$.

(b) Consider $w(x, t) := u(x, t) - u(1-x, t)$

$$\Rightarrow \left\{ \begin{array}{l} w_t = w_{xx} \\ w(x, 0) = 0 \\ w(0, t) = 0 = w(1, t) \end{array} \right. \begin{array}{l} \Rightarrow w \equiv 0, \\ \text{max princ i.e.} \\ u(x, t) = u(1-x, t) \end{array}$$

$$(c) \frac{d}{dt} \int_0^1 u^2 dx = 2 \int u u_t = 2 \int u u_{xx} = -2 \int u_x^2 \leq 0 \quad \square$$

Diffusion on the whole line

$$\boxed{u_t = u_{xx}} \quad t > 0, \quad x \in \mathbb{R}$$

fundamental solution

$$S(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-|x|^2/4t}$$

Prop (i) S is a solution,

$$\text{i.e. } S_t = S_{xx} \quad \forall x \in \mathbb{R}, \forall t > 0.$$

$$(ii) \int_{-\infty}^{\infty} S(x, t) dx = 1 \quad \forall t > 0.$$

$$(iii) S(x, t) \xrightarrow{t \rightarrow 0} \delta(x).$$

Proof (i) checked in 1st week ✓

(ii) $I := \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-|x|^2/4t} dx$

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{4\pi t} e^{-(x^2+y^2)/4t} dx dy$$

$$\Rightarrow \int_0^\infty \int_0^{2\pi} \frac{1}{4\pi t} e^{-r^2/4t} r d\varphi dr$$

$$x = r \cos \varphi$$

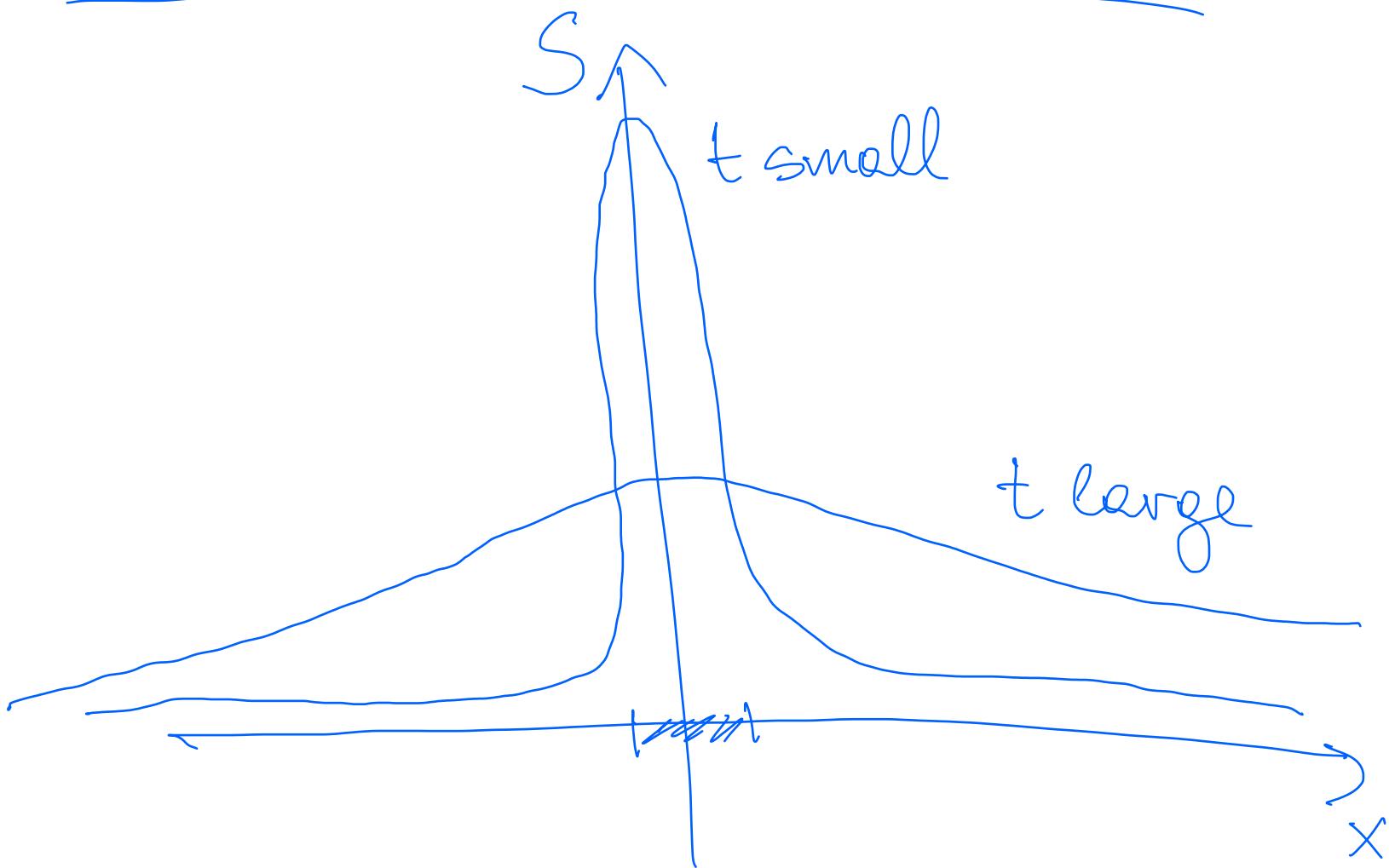
$$y = r \sin \varphi$$

$$= \int_0^\infty e^{-r^2/4t} \frac{r}{2t} dr$$

$$= \left[-e^{-r^2/4t} \right]_0^\infty = 1$$

□

How does S look like?



Q: What happens for $t \downarrow 0$?

$$S(x, t) \xrightarrow{t \downarrow 0} \delta(x), \text{ i.e.}$$

$$\int_{-\infty}^{\infty} S(x, t) \phi(x) dx \xrightarrow{t \downarrow 0} \phi(0).$$

(TODO: Look up Dirac delta, Gaussian)

Then

$$\left\{ \begin{array}{ll} u_t = u_{xx} & t > 0, x \in \mathbb{R} \\ u(x, 0) = \phi(x) \end{array} \right.$$

has the solution

$$u(x, t) = \int_{-\infty}^{\infty} S(x-y, t) \phi(y) dy$$

Proof (i) $(u_t - u_{xx})(x, t)$

$$= \int_{-\infty}^{\infty} (S_t(x-y, t) - S_{xx}(x-y, t)) \phi(y) dy$$

$\underbrace{\qquad\qquad}_{\Rightarrow 0}$

$$= 0.$$

(ii) $\lim_{t \downarrow 0} u(x, t) = \int_{-\infty}^{\infty} \delta(x-y) \phi(y) dy = \phi(x)$

□

$$S(x-y, t) \xrightarrow{t \downarrow 0} \delta(x-y) \equiv \delta_x(y)$$

Exer Solve $\begin{cases} u_t = u_{xx} & t > 0, x \in \mathbb{R} \\ u(x, 0) = e^{-x} \end{cases}$

$$u(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} e^{-y} dy$$

$$\frac{(x-y)^2}{4t} + y = \frac{x^2 - 2xy + y^2 + 4ty}{4t}$$

$$= \frac{(y+2t-x)^2}{4t} - t + x$$

$$\Rightarrow u(x, t) = e^{t-x} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(y+2t-x)^2}{4t}} dy = 1$$

$$\Rightarrow u(x, t) = e^{t-x}$$

Midterm: Mo Feb 8 (during class)