A GLIMPSE AT NONLINEAR PDES

LECTURE NOTES FOR APRIL 5

1. Dirichlet principle revisited

In the previous lecture we considered the Dirichlet problem for the Laplace equation,

\[
\begin{cases}
\Delta u = 0 & \text{in } D, \\
u = h & \text{on } \partial D,
\end{cases}
\]

where, \( D \subset \mathbb{R}^3 \) is the given domain, and \( h : \partial D \to \mathbb{R} \) is a given function. We learned that the solution minimizes the Dirichlet energy

\[
E[u] = \frac{1}{2} \int_D |\nabla u|^2.
\]

Namely, we proved:

**Theorem 1.3** (Dirichlet principle). Let \( u \) be the unique solution of the Dirichlet problem (1.1). If \( w \) is any other function in \( D \) satisfying \( w = h \) on \( \partial D \), then

\[
E[w] \geq E[u].
\]

In other words, our harmonic function \( u \) minimizes the energy among all functions (with the same boundary condition).

Here is another way to think about this: Since \( u \) minimizes the energy functional \( E \), it is in particular a critical point of \( E \). Thus, for any test function \( v : D \to \mathbb{R} \) that vanishes on \( \partial D \) we have

\[
\frac{d}{d\varepsilon}|_{\varepsilon=0} E[u + \varepsilon v] = 0.
\]

To use this, let us explicitly compute the first variation of the energy functional:

\[
\frac{d}{d\varepsilon}|_{\varepsilon=0} E[u + \varepsilon v] = \frac{d}{d\varepsilon}|_{\varepsilon=0} \left( \frac{1}{2} \int_D |\nabla(u + \varepsilon v)|^2 \right)
\]

\[
= \int_D \langle \nabla v, \nabla u \rangle
\]

\[
= -\int_D v \Delta u,
\]
where we integrated by parts in the last step. Hence, we obtain

$$\int_D v \Delta u = 0.$$  

Since this holds for all test functions $v$ as above, we conclude that

$$\Delta u = 0 \quad \text{in } D.$$  

To recap, we have shown that whenever $u$ is a critical point of $E$, then $u$ solves the PDE (1.10).

Finally, let us observe since the Dirichlet energy $E$ is strictly convex, there is a unique critical point, and this critical point is a minimum.

### 2. Calculus of variations and nonlinear PDEs

Many PDEs in mathematics, physics, engineering, economics, etc, arise as critical points of some energy functional. There is a big area in mathematics, called the calculus of variations, which systematically studies such problems. Let us consider some examples:

**Example 2.1** (Nonlinear Laplace equation). Consider the energy functional

$$E[u] = \int_D \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} u^{p+1} \right).$$

Let us compute the first variation:

$$\frac{d}{d\varepsilon} \big|_{\varepsilon=0} E[u + \varepsilon v] = \int_D \left( \langle \nabla u, \nabla v \rangle - u^p v \right)$$

$$= -\int_D v (\Delta u + u^p).$$

Hence, the critical points of $E$ solve the nonlinear Laplace equation

$$\Delta u + u^p = 0.$$  

**Example 2.6** (Nonlinear wave equation). Consider the action functional

$$A[u] = \int_D \left( \frac{1}{2} (\partial_t u)^2 - \frac{1}{2} |\nabla u|^2 + \frac{1}{p+1} u^{p+1} \right),$$

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In physics, the *action* is the kinetic energy minus the potential energy.
where now $u$ also depends on time. The first variation is:

\[
\frac{d}{d\varepsilon}|_{\varepsilon=0} A[u + \varepsilon v] = \int_D \left( \partial_t u \partial_t v - \langle \nabla u, \nabla v \rangle + u^p v \right)
\]

\[
= \int_D \left( - \partial_t^2 u + \Delta u + u^p \right) v.
\]

Hence, the critical points of $A$ solve the nonlinear wave equation

\[
\partial_t^2 u = \Delta u + u^p.
\]

**Example 2.11** (Minimal surfaces). Another famous problem is to find the shape of a soap film that spans a given wire ring. When we dip the wire ring into the soap water, a ”minimal surface” will form, i.e. a surface that minimizes area subject to the condition that its boundary is given by the wire. The geometry is simplest when the surface can be written as graph of a function $z = u(x, y)$, where $(x, y) \in D$, and the wire ring is the graph of a function $h : \partial D \to \mathbb{R}$. We thus must minimize the area functional

\[
A[u] = \int_D \sqrt{1 + u_x^2 + u_y^2}
\]

among all functions $u : D \to \mathbb{R}$ satisfying the boundary condition $u|_{\partial D} = h$. We compute

\[
\frac{d}{d\varepsilon}|_{\varepsilon=0} A[u + \varepsilon v] = \int_D \frac{1}{\sqrt{1 + u_x^2 + u_y^2}} \left( u_x v_x + u_y v_y \right).
\]

Hence, after integration by parts, we see that critical points of the area functional satisfy the minimal surface equation

\[
\left( \frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}} \right)_x + \left( \frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}} \right)_y = 0.
\]

The minimal surface equation shares some similarities with the Laplace equation $u_{xx} + u_{yy} = 0$, but is quite a bit more complicated due to its nonlinear nature. It is one of the most classical and well studied geometric PDEs\footnote{In case you are wondering what your professor is doing most of the time: I study the corresponding evolution equation, called the mean curvature flow.}. You can google “minimal surface” and look at some beautiful pictures. And if you have solved the bonus question from the last problem set, then you have already seen the catenoid.
3. DIRICHLET PROBLEM FOR THE NONLINEAR LAPLACE EQUATION

Let us now consider the Dirichlet problem for the nonlinear Laplace equation:

\[
\begin{cases}
\Delta u + u^p = 0 & \text{in } D, \\
u = h & \text{on } \partial D.
\end{cases}
\]

Here, \( p > 1 \) is a given number. Note that \( \Delta u + u^p = 0 \) is a nonlinear equation, i.e. the sum of two solutions is not a solution in general. This nonlinearity gives rise to new interesting phenomena in stark contrast to the linear analysis from the previous section. In particular, we will see that the behaviour of (3.1) depends crucially on the exponent \( p \).

**Example 3.2** (radial solutions). Suppose \( D = B_1(0) \) is the unit ball, \( p = 5 \), and \( h \equiv \sqrt{\frac{3}{2}} \). Let us look for a radial solution \( u = u(r) \). Recalling the formula for the Laplacian in spherical coordinates, we thus have to solve the ODE

\[
u_{rr} + \frac{2}{r} u_r + u^5 = 0.
\]

Let us make the ansatz

\[
u = \frac{c}{\sqrt{1 + r^2}}.
\]

Differentiating gives

\[
u_r = -\frac{cr}{(1 + r^2)^{3/2}},
\]

and

\[
u_{rr} = \frac{3cr^2}{(1 + r^2)^{5/2}} - \frac{c}{(1 + r^2)^{3/2}}.
\]

Thus,

\[
u_{rr} + \frac{2}{r} u_r + u^5 = \frac{3cr^2}{(1 + r^2)^{5/2}} - \frac{3c}{(1 + r^2)^{3/2}} + \frac{c^5}{(1 + r^2)^{5/2}}
\]

\[
= \frac{-3c + c^5}{(1 + r^2)^{5/2}}.
\]

Hence, choosing \( c = 3^{1/4} \) we have found a solution. \( \square \)

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3Note that for \( r \) large the function \( u \) approaches \( v = \frac{c}{r} \) which solves \( v_{rr} + \frac{2}{r} v_r = 0 \).
Recall that associated to the problem \((3.1)\) we have the energy functional
\[
E[u] = \frac{1}{2} \int_D |\nabla u|^2 - \frac{1}{p+1} \int_D |u|^{p+1}.
\]
For \(p\) close to 1 the term \(\int_D |\nabla u|^2\) dominates. For \(p\) very large the term \(\int_D |u|^{p+1}\) dominates. The transition happens at the critical exponent \(p = 5\) that we have encountered in our example. In fact:

**Theorem 3.10** (existence vs nonexistence). The problem
\[
\begin{cases}
\Delta u + u^p = 0 & \text{in } B, \\
u = 0 & \text{on } \partial B.
\end{cases}
\]
has nontrivial solutions for \(1 < p < 5\), but only the zero solution for \(p > 5\).

**Sketch of proof.** If \(1 < p < 5\) one can find nontrivial solutions via the mountain pass method (we skip this, but if you are interested you can read it in the book by Evans in Section 8.5).
Suppose now \(p > 5\). Multiplying our equation by \(u\) and integrating by parts we obtain
\[
\int_B |\nabla u|^2 = \int_B |u|^{p+1}.
\]
On the other hand, multiplying our equation by \(x \cdot \nabla u\) and integrating we obtain
\[
\int_B \Delta u x \cdot \nabla u + \int_B u^p x \cdot \nabla u = 0.
\]
After integration by parts (we also skip this remarkable computation, but if you are interested you can read it in the book by Evans in Section 9.4) this can be rewritten as
\[
\int_B |\nabla u|^2 + \int_{\partial B} |x||\nabla u|^2 = \frac{6}{p+1} \int_B |u|^{p+1}.
\]
Subtracting \((3.12)\) from \((3.14)\) we obtain
\[
\left(\frac{6}{p+1} - 1\right) \int_B |u|^{p+1} \geq 0.
\]
Since \(p > 5\) we have \(\frac{6}{p+1} - 1 < 0\), hence \(u \equiv 0\). \(\Box\)