MATD46 Assignment 3, due Feb 28

1. No maximum principle for the wave equation

Show that there is no maximum principle for the wave equation (Hint: Turn the argument that we sketched in class into a detailed proof).

2. Diffusion on the half-line

Solve $u_t = u_{xx}, u(x,0) = e^{-x}, u(0,t) = 0$ on the half-line $0 < x < \infty$ (Hint: Read about the reflection trick in Sec. 3.1).

3. A metal rod

Consider a metal rod ($0 < x < \ell$), insulated along its sides but not at its ends, which is initially at temperature equal to 1. Suddenly both ends are plunged into a bath of temperature 0. Write down a PDE model including boundary conditions and initial conditions. Find a formula for the temperature $u(x,t)$ at positive times. In this problem, you can use the fact that

$$\frac{\pi}{4} = \sin\frac{\pi x}{\ell} + \frac{1}{3}\sin\frac{3\pi x}{\ell} + \frac{1}{5}\sin\frac{5\pi x}{\ell} + \ldots.$$

4. Waves in a resistant medium

Consider waves in a resistant medium that satisfy the problem

$$\begin{aligned} u_{tt} = c^2 u_{xx} - r u_t \quad &\text{for } 0 < x < \ell, \\ u = 0 \quad &\text{at both ends}, \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x).& \end{aligned}$$

Here, r is a constant with $0 < r < 2\pi c/\ell$. Find a series expansion of the solution.

5. Particle in a box

A quantum-mechanical particle on the line with an infinite potential outside the interval $(0,\ell)$, i.e. “a particle in a box”, is described by Schrödinger’s equation $u_t = iu_{xx}$ on $(0,\ell)$ with Dirichlet boundary conditions. Use separation of variables to find a series representation for the solution (with coefficients depending on the initial condition).

6. Mixed boundary conditions

Solve the diffusion problem $u_t = u_{xx}$ in $0 < x < \ell$, with the mixed boundary conditions $u(0,t) = 0 = u_x(\ell,t)$.

7. Periodic boundary conditions

Consider diffusion inside an enclosed circular tube. Let its length (circumference) be 2ℓ. Let x denote the arc length parameter where $-\ell \le x \le \ell$. Then the concentration of the diffusing substance is modelled by the equation

$$u_t = u_{xx} \quad \text{for } -\ell \le x \le \ell,$$

with the periodic boundary conditions

$$u(-\ell,t) = u(\ell,t) \text{ and } u_x(-\ell,t) = u_x(\ell,t).$$

Use separation of variables to find a series representation for the solution (with coefficients depending on the initial condition).

Please feel free to discuss the homework problems among yourselves and with me. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

We will randomly select 3 questions, for which you will receive points $p_1, p_2, p_3 \in \{0,1,2,3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1+p_2+p_3-s, 0) \in \{0,1,\ldots,9\}$.