MATD46 Assignment 1, due Jan 24

1. Harmonic functions

Consider the space V of all harmonic functions in the plane that grow at most quadratically. Namely, let

$$V := \left\{ u : \mathbb{R}^2 \to \mathbb{R} \,|\, u_{xx} + u_{yy} = 0 \text{ and } \sup_{x,y} \frac{|u(x,y)|}{1 + x^2 + y^2} < \infty \right\}.$$

i. Prove that V is a vector space.

ii. Try to find all elements of V.

2. Inhomogenous constant coefficient equation

Solve the equation $2u_x + 3u_y = 4u$.

3. Method of characteristics

i. Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$.

ii. In which region of the plane is the solution uniquely determined?

4. Ill-posed problem

Consider the equation

$$u_x + yu_y = 0$$

with the boundary condition $u(x, 0) = \phi(x)$.

i. For $\phi(x) = x$, show that no solution exists.

ii. For $\phi(x) = 1$, show that there are many solutions.

5. Heat equation and subsolution

Suppose $u = u(x,t)$ solves the heat equation $u_t - u_{xx} = 0$. Consider the quantity $v := u^2 + 2tu_x^2$. Show that v is a subsolution, i.e. show that $v_t - v_{xx} \leq 0$.

6. Deriving a PDE model

A flexible chain of length ℓ is hanging from one end $x = 0$ but oscillates horizontally. Let the x-axis point downward and the u-axis point to the right. Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain. Assume that the oscillations are small. Find the PDE satisfied by the chain.

7. Schrödinger equation

Consider a solution $u = u(x, y, z, t)$ of the Schrödinger equation

$$iu_t = -\Delta u + Vu.$$

Here $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ denotes the Laplacian, and $V = V(x, y, z)$ is a given real-valued function called the potential. Show that if $\int |u|^2 = 1$ at $t = 0$, then $\int |u|^2 = 1$ for all t. (You may assume that all quantities involved are sufficiently smooth and decay sufficiently fast at infinity).

Please feel free to discuss the homework problems among yourselves and with me. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

We will randomly select 3 questions, for which you will receive points $p_1, p_2, p_3 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 + p_3 - s, 0) \in \{0, 1, \ldots, 9\}$.