Introduction to Real Analysis (MAT C37), Practice Problems

1. Review of some key definitions and theorems

State the following definitions and theorems (make sure that you can give clear answers if I ask you something like that at the final)

- Definition of the Lebesgue measure
- Definition of Lebesgue measurable
- Definition of the Lebesgue integral
- Dominated convergence theorem
- Monotone convergence theorem
- Fatou's lemma
- Definition of L^1 and L^2
- Fundamental theorem of Calculus

2. Space of integrable and square integrable functions

- Is the function $f(x) = \frac{x}{1+x^2}$ in $L^1(\mathbb{R})$?
- Give an example of a function that is in $L^1(\mathbb{R})$ but not in $L^2(\mathbb{R})$.
- Prove that $L^2([a,b]) \subseteq L^1([a,b])$.

3. Interchanging limits and integration

Compute the following limits (here and everywhere else you should justify your steps):

- $\lim_{n\to\infty} \int_0^1 \frac{1}{1+x^n}\, dx$.
- $\lim_{n\to\infty} \int_0^\infty \frac{1+\cos\left(\frac{x}{n}\right)}{2-\sin\left(\frac{x}{n}\right)} \frac{1}{1+x^2}\, dx$.
- $\lim_{n\to\infty} \int_0^\infty \frac{x^2+n^2}{x^2+2n^2} e^{-x}\, dx$
- $\lim_{n\to\infty} \int_{-n}^n \frac{\cos(x/n)}{(1+x^2)}\, dx$

4. Lebesgue measurable and Lebesgue measure

Consider the set $E := \tilde{C} \cup [7,9] \subseteq \mathbb{R}$, where $\tilde{C}$ denotes the fat Cantor set where at each step a middle quarter is removed.

- Prove that E is Lebesgue measurable.

- Compute the Lebesgue measure of E.

5. A parameter dependent integral

The goal of this question is to compute the integral $\int_0^\infty \cos(tx)e^{-x^2/2}dx$.

- Prove that $F(t) := \int_0^\infty \cos(tx)e^{-x^2/2}dx$ is differentiable.
- Compute $F(t)$. (Hint: Derive a differential equation for $F(t)$ and solve it.)