# Practice Problems (Term Test from last year)

## 1. Exterior Lebesgue measure

i. What is the definition of the exterior Lebesgue measure?

ii. Let $C \subseteq \mathbb{R}$ be the Cantor set, $\mathbb{Q} \subseteq \mathbb{R}$ the rational numbers, and $[1, 43] \subseteq \mathbb{R}$ the closed interval of all real numbers between 1 and 43. Compute $m_*(C \cup \mathbb{Q} \cup [1, 43])$.

## 2. Lebesgue measurability and Lebesgue measure

For any set $A \subseteq \mathbb{R}$ we consider the set

$$A' := \{-x \mid x \in A\},$$

which is obtained by reflecting $A$ across the origin.

i. Prove that $A$ is measurable if and only if $A'$ is measurable.

ii. Assume now that $A$ is measurable (by the first part this implies that $A'$ is also measurable). Prove that $m(A) = m(A')$.

## 3. Borel sets

i. What is the definiton of a $G_\delta$ set?

ii. Answer with yes or no: The unit cube $[0, 1]^2 \subseteq \mathbb{R}^2$ is open? closed? $F_\sigma$? $G_\delta$?

## 4. Topological smallness vs. measure theoretic smallness

A subset $A \subseteq \mathbb{R}$ is called *nowhere dense* if the interior of its closure is empty.

i. Prove or disprove: If $m_*(A) = 0$, then $A$ is nowhere dense.

ii. Prove or disprove: If $A$ is nowhere dense, then $m_*(A) = 0$.