MATC37 Assignment 3, due Feb 28

1. Measurability of $-f$ and f^2

Let $f : \mathbb{R}^d \to \mathbb{R}$ be measurable.

i. Prove that $-f$ is measurable.

ii. Prove that f^2 is measurable.

2. Measurability of some special functions

i. Prove that the function $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin(1 + e^x) + \frac{1}{1+x^2}$ is measurable.

ii. Prove that every monotone function $f : \mathbb{R} \to \mathbb{R}$ is measurable.

3. Further characterizations of measurable functions

Let $f : \mathbb{R}^d \to \mathbb{R}$.

i. Prove that f is measurable if and only if for all $q \in \mathbb{Q}$ the set $\{f < q\}$ is measurable.

ii. Prove that f is measurable if and only if for every compact $C \subseteq \mathbb{R}$ the preimage $f^{-1}(C)$ is measurable.

4. Riemann integrability is not preserved under limits

Let $f : [0,1] \to \mathbb{R}$ be defined by $f = \chi_{\mathbb{Q}\cap[0,1]}$, i.e. $f(x) = 1$ for $x \in \mathbb{Q} \cap [0,1]$, and $f(x) = 0$ for $x \in [0,1] \setminus \mathbb{Q}$.

i. Show that f is not Riemann integrable (you may wish to look up the definition of Riemann integrable in some lecture notes from your previous courses or in a textbook)

ii. Construct a sequence $f_n : [0,1] \to \mathbb{R}$ of Riemann integrable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0,1]$.

5. Riemann integrability is not even preserved under monotone limits of continuous functions

This exercise provides a construction of a decreasing sequence of positive continuous functions on the unit interval $[0,1]$ whose pointwise limit is *not* Riemann integrable. Let $\widetilde{C} \subseteq [0,1]$ be a Cantor-like set with $m(\widetilde{C}) > 0$. (recall from the previous problem set that in the construction of $\widetilde{C}$ at the k-th step one removes 2^{k-1} centrally situated open intervals of lenght ℓ_k. By choosing the numbers $\ell_1, \ell_2, \dots$ small enough such that $\sum_{k=1}^{\infty} 2^{k-1}\ell_k < 1$ we can ensure that $m(\widetilde{C}) = 1 - \sum_{k=1}^{\infty} 2^{k-1}\ell_k > 0$).

Let F_1 denote a piece-wise linear and continuous function on $[0,1]$ with $F_1 = 1$ in the complement of the removed interval from the first step, $F_1 = 0$ at the center of the removed interval, and $0 \le F_1(x) \le 1$ for all x. Similarly, construct $F_2 = 1$ in the complement of the intervals from step two with $F_2 = 0$ at the center of the removed intervals from step two, and $0 \le F_2 \le 1$, etc (see also Figure 5 on page 40 of the textbook).

Let $f_n := F_1 \cdots F_n$. Prove the following:

i. Show that for all $n \geq 1$ and all $x \in [0,1]$ we have $0 \leq f_n(x) \leq 1$ and $f_{n+1}(x) \leq f_n(x)$, and use this to infer that $f(x) = \lim_{n\to\infty} f_n(x)$ exists.

ii. Show that the function f is discontinuous at every point of $\widetilde{C}$. (Hint: Note that $f(x) = 1$ for $x \in \widetilde{C}$, and find a sequence of points x_k so that $x_k \to x$ and $f(x_k) = 0$)

iii. Conclude that the function f is not Riemann integrable (You may use the following fact without proving it: A bounded function is Riemann integrable if and only if its set of discontinuities has measure zero.)

Please feel free to discuss the homework problems among yourselves and with me. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

We will randomly select 3 questions, for which you will receive points $p_1, p_2, p_3 \in \{0,1,2,3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 + p_3 - s, 0) \in \{0, 1, \ldots, 9\}$.