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CHAPTER 1

Introduction

These notes are a work in progress and are probably full of typos and other little mistakes. If you notice any, please send email pointing them out to rjerrard@math.toronto.edu.

1. Goals

This class aims to examine mathematical questions related to the flow of fluids with concentrated vorticity. We will focus on

- inviscid incompressible fluids, as described by the Euler equations

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= 0 \\
\nabla \cdot u &= 0
\end{align*}
\]

for \( u : \mathbb{R}^N \times [0, T) \to \mathbb{R}^N \) and \( p : \mathbb{R}^N \times [0, T) \to \mathbb{R} \), for \( N = 2 \) or \( 3 \); and

- quantum mechanical fluids, as described by the Gross-Pitaevskii equation

\[
\partial_t \psi - \frac{1}{\epsilon^2} (|\psi|^2 - 1) \psi = 0, \quad 0 < \epsilon \ll 1
\]

for \( \psi : \mathbb{R}^N \times [0, T) \to \mathbb{C} \). One may think of the limit \( \epsilon \to 0 \) as the incompressible limit; in this regime, there are good heuristic reasons, rigorously verified in some situations, to expect similarities between (1) and (2).

We may also consider variants such as problems on bounded domains or in higher dimensions.

We will be particularly interested in flows that possess structures such as vortex rings or (more generally) vortex filaments. Without yet defining these terms, one can find numerous videos online that purport to show examples of these things, in both classical and quantum mechanical fluids:

- vortex rings in air
- vortex rings in water
- more general vortex filaments in water
- another example of vortex filaments in water
- vortex filaments in superfluid helium

Equations (1) and (2) described (idealized versions of) fluids such as water, air, liquid helium etc. We may therefore hope to prove the existence of solutions of these equations that exhibit the behaviours seen in the videos. This leads to questions such as these:

Do these equations have solutions with concentrated vortex filaments that persist over a reasonable time interval?

If so, are there laws of motion that govern how these filaments evolve?
What should these laws be, and can they be rigorously verified?

This class will focus on these questions. In particular, we will study the following phenomena, listed in roughly increasing order of difficulty.

1. point vortices in 2 dimensions (where here “point vortex” means a region of very small size with a $O(1)$ amount of vorticity). We will consider
   - configurations of point vortices that evolve by rigid motion such as rotation or translation. These give rise to either elliptic PDEs or variational problems.
   - dynamics of point vortices in more general situations.

2. vortex rings in 3-d, by which we mean concentrations of vorticity that exhibit cylindrical symmetry. Writing equations in cylindrical $(r, \theta, z)$ coordinates, these are problems that reduce to ones involving only the $(r, z)$ variables. As above, we may ask about
   - a single ring that evolves by rigid motion, without change of form, as seen in some of the videos above. Again, these are described by elliptic or variational problems
   - genuinely dynamical problems, such as
     - a single vortex ring, but one that does not evolve by rigid motion
     - multiple vortex rings, far enough from each other that interaction between them can be neglected.
     - multiple vortex rings with significant interaction between them. This gives rise to the phenomenon of leapfrogging vortex rings, first described qualitatively by Helmholtz in the 1850s. [This can be seen in actual fluids.]

3. vortex filaments in 3-d that are small perturbations of straight lines. A common example arises from the fact that airplane wings shed vortices as they pass through air, leaving behind them a pair of filaments that are approximately linear (if the airplane is not turning dramatically).

![Figure 1. from a video filmed by D. Smets](image)

As above, one can look for nearly linear filaments that evolve by rigid motion, giving rise to elliptic or variational problems, or one can consider the full dynamical problem.
One can view 2-d point vortices as corresponding to 3-d fluid flows with translational symmetry in the z direction, and such that the velocity is purely horizontal. Nearly rectilinear vortex filaments may be thought of as possessing approximate translation-invariants in the z-direction.

(4) general smooth vortex filaments in 3-d such as those seen in some of the videos above, without any symmetries or approximate symmetries.

2. Quick introduction to the Euler equations

We consider a classical incompressible fluid of constant density, flowing without friction in a connected open set \( \mathcal{D} \subset \mathbb{R}^N \). We will generally focus on \( \mathcal{D} = \mathbb{R}^N \) and \( N = 2 \) or 3.

We will typically assume that we know the state of the fluid at time \( t = 0 \), and we want to study its motion for \( t \in (0, T) \), where \( T \) may be \(+\infty\).

Here we briefly summarize some basics, omitting most proofs of standard facts. We will focus on issues relevant to our main themes. Readers who have never seen these facts before are encouraged to verify some or all of them as exercises; see Section 2.9. The topics we discuss in more detail are often less important, in the big picture, than the ones whose proofs we omit but are relevant to our later concerns.

2.1. Eulerian and Lagrangian perspectives. Before considering any equations of motion, we briefly discuss the issue of how to describe the motion of a fluid. For the time being we largely ignore any considerations related to smoothness of the fluid flow – we will just assume that the objects we consider are smooth enough to easily justify all computations – and we use words like “diffeomorphism” in an imprecise sense.

2.1.1. Lagrangian description. One way to describe fluid motion is via a map

\[(\alpha, t) \in \mathcal{D} \times (0, T) \mapsto X(\alpha, t) \in \mathcal{D}\]

interpreted as

\[X(\alpha, t) = \text{position at time } t \text{ of the “fluid particle” initially located at } \alpha.\]

With this in mind, we will refer to a \( \mathcal{C}^k \) particle trajectory map to indicate a map \( X : \mathcal{D} \times [0, T) \to \mathcal{D} \) such that

- \( X(\alpha, 0) = \alpha \) for every \( \alpha \in \mathcal{D} \)
- \( \alpha \mapsto X(\alpha, t) \) is a \( \mathcal{C}^k \) diffeomorphism \( \mathcal{D} \to \mathcal{D} \) for every \( t \in (0, T) \)

Obviously we can replace \( \mathcal{C}^k \) by other measures of smoothness, if we wish.

2.1.2. Eulerian description. In this description, the fluid motion is described in terms of its velocity, given by a map

\[(x, t) \in \mathcal{D} \times (0, T) \mapsto u(x, t) \in \mathbb{R}^N\]

interpreted as

\[u(x, t) = \text{velocity of the fluid at position } x \text{ and time } t.\]

We will sometimes write a velocity field \( u \) in components as \( u = (u^1, \ldots, u^n) \). We may also use other letters, such as \( v, w, \ldots \) to denote a velocity field.

\(^1\) that is, not a quantum mechanical fluid
If we are given a sufficiently smooth particle trajectory map, then the associated velocity field is defined by

\[
\frac{dX}{dt}(\alpha, t) = u(X(\alpha, t), t).
\]

Conversely, given a smooth enough (for example Lipschitz continuous) velocity field \( u \), an associated particle trajectory map can be found by solving the ODE with the initial data \( X(\alpha, 0) = \alpha \). Standard ODE results guarantee that the map \( X \) generated in this way is a particle trajectory map, as long as \( u \cdot \nu = 0 \) on \( \partial \mathcal{D} \), where \( \nu \) denotes the outer unit normal.

The relationship between the Lagrangian and Eulerian descriptions can be subtle if the fluid flow is not very smooth.

2.2. Some physical quantities. We now consider several basic notions, and we will see the form they take in both the Lagrangian and the Eulerian descriptions. Throughout the following discussion, we assume for convenience that \( X : \mathcal{D} \times [0, T) \to \mathcal{D} \) is a \( C^2 \) particle trajectory map describing a fluid flow, and that the corresponding velocity field is \( u : \mathcal{D} \times [0, T) \to \mathbb{R}^N \). These assumptions can often be weakened.

In this situation, the fluid acceleration is defined by

\[
\left( \frac{\partial}{\partial t} \right)^2 X(\alpha, t) := \text{the acceleration at time } t \text{ and position } X(\alpha, t).
\]

A simple calculation then shows that in the Eulerian description,

\[
\partial_t u(x, t) + (u \cdot \nabla) u(x, t) = \text{acceleration at time } t \text{ and position } x,
\]

where \( u \cdot \nabla \) is the differential operator \( u^i \partial_j \)

We may continue to refer to and think of \( \partial_t u + u \cdot \nabla u \) as the Eulerian expression for acceleration even when the smoothness assumptions above are not satisfied.

Next, the flow described by \( X \) is said to be incompressible if

\[
\text{det } \nabla X(\alpha, t) = 1 \quad \text{for all } (\alpha, t).
\]

It is a fact that

\[
\text{the flow is incompressible } \iff \mathcal{L}^n(\mathcal{O}) = \mathcal{L}^n(X(\mathcal{O}, t))
\]

for all open \( \mathcal{O} \subset \mathcal{D} \) and \( t \in (0, T) \),

\[
\nabla \cdot u(x, t) = 0 \quad \text{for all } (x, t) \in \mathcal{D} \times [0, T).
\]

where \( X(\mathcal{O}, t) := \{ X(\alpha, t) : \alpha \in \mathcal{O} \} \). We prove a slightly more version of this fact at the end of this section.

Next, given a \( C^1 \) function \( f : \mathcal{D} \times [0, T) \to \mathbb{R} \) is \( C^1 \), we define

the material derivative of \( f \) at \( (X(\alpha, t), t) \) is defined as \( \frac{d}{dt} f(X(\alpha, t), t) \).

---

\[2 \text{ Throughout most of these notes, we will implicitly sum over repeated indices, so that for example } u_j \partial_j \text{ will be understood to mean } \sum_{j=1}^N u_j \partial_j = u \cdot \nabla, \text{ and } (u \cdot \nabla)u \text{ is the vector field whose } i\text{th component is } \sum_{j=1}^N u^i \partial_j u^j. \]
This measures the rate of change of the quantity $f$ along a particle trajectory $t \mapsto X(\alpha, t)$. We will write $\frac{Df}{Dt}$ to denote the material derivative in the Eulerian picture. An easy computation shows that

$$\text{material derivative of } f = \frac{Df}{Dt} = (\partial_t + u \cdot \nabla)f.$$

Additional physical quantities are discussed in Section 2.8 below.

The following lemma implies (4). The extra generality may be useful later.

**Lemma 1.** Assume that $D \subset \mathbb{R}^N$ is connected and open and that $b : D \rightarrow (0, \infty)$ is $C^1$. Let $X : D \times [0, T) \rightarrow D$ be a $C^2$ particle trajectory map and $u : D \times [0, T) \rightarrow \mathbb{R}^N$ be the associated velocity vector field. Then the following are equivalent:

1. $\nabla \cdot (bu) = 0$ in $D \times [0, T)$,
2. $\det \nabla X(\alpha, t) = \frac{b(\alpha)}{b(X(\alpha, t))}$,
3. For all open $O \subset D$ and $t \in [0, T)$, $\int_{X(\alpha, t)} b(x) \, dx = \int_O b(x) \, dx$.

We state part of the proof as a separate lemma.

**Lemma 2.** For open $D \subset \mathbb{R}^N$, if $X : D \times [0, T) \rightarrow D$ is a particle trajectory map and $u : D \times [0, T) \rightarrow \mathbb{R}^N$ is the associated velocity field, then

$$\frac{d}{dt} \det \nabla X(\alpha, t) = \nabla \cdot (u(x, t)|_{x=X(\alpha, t)}) \det \nabla X(\alpha, t).$$

**Proof.** First we compute $\frac{d}{dt}(\det \nabla X(\alpha, t))$. In general if $M(t)$ is a $N \times N$ matrix-valued function of $t$, then for any $t_0$,

$$\left. \frac{d}{dt} M(t) \right|_{t=t_0} = \left. \frac{d}{dt} (M(t_0)M(t_0)^{-1}M(t)) \right|_{t=t_0} = \det M(t_0) \left. \frac{d}{dt} (M(t_0)^{-1}M(t)) \right|_{t=t_0}.$$

Also, for $N(t) = M(t_0)^{-1}M(t)$, since $N(t_0) = \text{Id}$, it is easy to see that $\left. \frac{d}{dt} N(t) \right|_{t=t_0} = \text{Tr}(N'(t_0))$. Thus

$$\left. \frac{d}{dt} M(t) \right|_{t=t_0} = \det M(t_0) \text{Tr}(M^{-1}(t_0)\partial_t M(t_0))$$

For $M(t) = \nabla X(\alpha, t)$,

$$\partial_t \nabla X(\alpha, t) = \nabla \partial_t X(\alpha, t) = \nabla u(x, t)|_{x=X(\alpha, t)}$$

and

$$\nabla X)^{-1}(\alpha, t) = \nabla x(\alpha X(\alpha, t), t)$$
where we are writing $\alpha(x, t)$ to denote the inverse of $X(\alpha, t)$, defined by $X(\alpha(x, t), t) = x$. Thus in terms of components,
\[
\operatorname{Tr} \left( \nabla X^{-1}(\alpha, t_0) \partial_t \nabla X(\alpha, t_0) \right) = \sum_{i,j} \left[ \partial_{\alpha_j} u^i(x, t) \frac{\partial \alpha^j}{\partial x_i}(x, t) \right]_{x=X(\alpha, t)} = \sum_i \partial_{x_i} u^i(x, t) \bigg|_{x=X(\alpha, t)} = \nabla \cdot u(x, t) \bigg|_{x=X(\alpha, t)}.
\]
The conclusion follows. \qed

**Proof of Lemma 1.** It follows from Lemma 2 that
\[
\partial_t \left[ b(X(\alpha, t)) \det \nabla \alpha X(\alpha, t) \right] = [\nabla b(X(\alpha, t)) \cdot \partial_t X(\alpha, t) + b(X(\alpha, t)) \nabla \cdot u(X(\alpha, t), t)] \det \nabla \alpha X(\alpha, t) = \nabla \cdot (bu) \bigg|_{x=X(\alpha, t)} \det \nabla \alpha X(\alpha, t)
\]
By assumption $X(\cdot, t)$ is a $C^2$ diffeomorphism for every $t$, so $\det \nabla \alpha X$ never vanishes. The above equation thus implies that
\[
\partial_t \left[ b(X(\alpha, t)) \det \nabla \alpha X(\alpha, t) \right] = 0 \quad \text{for every } \alpha \in \mathcal{D} \text{ and } t \in [0, T)
\]
\[\iff \nabla \cdot (bu) = 0 \quad \text{in } \mathcal{D} \times [0, T).
\]
Since the initial conditions for $X$ imply that $b(X(\alpha, 0)) = b(\alpha)$ and $\nabla \alpha X(\alpha, 0) = \text{Id}$, it follows from this that $1 \iff 2$.

Then for $\Omega \subset \mathcal{D}$ and $t \in [0, T)$ we change variables by setting $x = X(\alpha, t)$. Then
\[
\int_{X(\alpha, t)} b(x) \, dx = \int_{\Omega} b(X(\alpha, t)) \det \nabla \alpha X(\alpha, t) \, d\alpha.
\]
From this we easily deduce that
\[
\int_{X(\alpha, t)} b(x) \, dx = \int_{\Omega} b(\alpha) \, d\alpha \quad \text{for all } \alpha, t
\]
\[\iff b(X(\alpha, t)) \det \nabla \alpha X(\alpha, t) = b(\alpha) \quad \text{for all } \alpha, t,
\]
that is, $2 \iff 3$. \qed

**2.3. the Euler equations in velocity-pressure form.** We continue to assume that $\mathcal{D}$ is an open subset of $\mathbb{R}^N$.

The Euler equations describe the motion of an incompressible fluid of constant density that flows without friction. They may be written in the form

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= 0, \\
\nabla \cdot u &= 0,
\end{align*}
\]

in $\mathcal{D} \times (0, T)$. \hfill (5)

Here $p : \mathcal{D} \times (0, T) \to \mathbb{R}$ is a function called the *pressure*. The equations are supplemented by initial conditions (for $u$ only) and also, if $\partial \mathcal{D}$ is nonempty, by boundary conditions, typically

\[
u \cdot n = 0 \quad \text{on } \partial \mathcal{D} \times (0, T). \hfill (6)
\]

Roughly speaking, the equations state that

- the fluid acceleration is (minus) the gradient of the pressure, where
the pressure is whatever is needed to insure that $\nabla \cdot u = 0$, in other words that the associated flow is incompressible.

It is worth noting that the first equation can be written in divergence form:

$$
\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0
$$

where $\nabla \cdot (u \otimes u)$ is the vector whose $i$th component is $\partial_j (u_i u_j)$.

**Remark 1.** If $\phi \in C^1_c(D \times [0, T]; \mathbb{R}^N)$ satisfies $\nabla \cdot \phi = 0$, then it follows from (5) and some elementary computations (including integration by parts) that

$$
\partial_t \int_D u \cdot \phi \, dx = \int_D \partial_t \phi \cdot u + (u \cdot \nabla \phi) \cdot u \, dx = \int_D \frac{D\phi}{Dt} \cdot u \, dx.
$$

After integrating with respect to $t$, this becomes

$$
\int_D u \cdot \phi \, dx \bigg|_0^T = \int_0^T \int_D \partial_t \phi \cdot u + (u \cdot \nabla \phi) \cdot u \, dx \, dt = \int_0^T \int_D \frac{D\phi}{Dt} \cdot u \, dx \, dt.
$$

This identity can be used as the basis for a definition of weak solution – one can declare a vector field $u$ to be a weak solution (ignoring possible boundary conditions for now, if $\partial \Omega \neq \emptyset$) if it satisfies the above equality for every test vector field $\phi$ as above.

Such a definition of weak solution makes sense if $u \in L^2_{loc}(D \times [0, T]; \mathbb{R}^N)$, since then all integrands are guaranteed to be integrable.

On the other hand, if $u \not\in L^2_{loc}$ then no definition of weak solution based on this identity can make sense, since then the integrand $(u \cdot \nabla \phi) \cdot u$ in general the integrand fails to be integrable. In fact, I do not know of any way of making sense of the Euler equations for velocity fields not in $L^2_{loc}$.

### 2.4. Vorticity in 3 dimensions.

If $N = 3$, then the *vorticity* is the vector field, denoted $\omega = \omega(x, t)$ defined by

$$
\omega := \nabla \times u.
$$

Vorticity has a number of important properties:

1. It is (sometimes) literally visible in fluid flows; you can see it on youtube, as in the examples in Section [1]

2. Under reasonable hypotheses, one can recover the velocity field from the vorticity.

3. The Euler equations admit an elegant reformulation in terms of the vorticity, in which the pressure does not appear.

4. This reformulation has certain remarkable consequences.

One can easily take the curl of the Euler equations to obtain an equation for the vorticity field $\omega$, in which the velocity field $u$ appears in the coefficients. In situations where the operator $\mathcal{K}$ is well-defined, this equation, together with the equation $u = \mathcal{K} \omega$ that determines $u$ from $\omega$, constitute (at least formally) a closed system of equations

$$
\partial_t \omega + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u
$$

This equation has a number of consequences:
1. Introduction

- The vorticity transport formula states that if \( u, \omega \) are smooth enough solutions of \( (9) \) and \( X \) is the associated particle trajectory map, then

\[
\omega(X(\alpha, t), t) = \nabla_\alpha X(\alpha, t) \omega^0(\alpha)
\]

where \( \omega^0(\alpha) = \omega(\alpha, 0) = \nabla \times u^0(\alpha) \) is the initial vorticity.

- A vortex line (at time \( t \)) is defined to be an integral curve of the vector field \( \omega(\cdot, t) : D \to \mathbb{R}^3 \). Thus, a vortex line is a curve of the form

\[
C = \{ c(s) : s \in I \}, \quad \text{for some } c : I \to \mathbb{R}^3 \text{ solving } \frac{d c}{d s} = \omega(c(s), t)
\]

where \( I \) is an interval. A consequence of \( (10) \) is that vortex lines are transported by the flow, in the sense that if \( C_0 \) is a vortex line at time \( t = 0 \), then \( C_t := \{ X(\alpha, t) : \alpha \in C_0 \} \) is a vortex line at time \( t \).

2.5. The Biot-Savart Law. In order to recover the velocity field from the vorticity, we have to solve the problem

\[
\begin{aligned}
\nabla \cdot \omega &= 0, \\
\nabla \times u &= \omega,
\end{aligned}
\]

We will discuss a number of variants of this problem later on. Now for simplicity we suppose that \( D \) is all of \( \mathbb{R}^3 \), and we assume that \( \omega \) is smooth, with rapid decay at infinity. In this situation we may proceed as follows:

1. First find \( \psi : \mathbb{R}^3 \to \mathbb{R}^3 \) such that \( -\Delta \psi = \omega \), and such that \( \psi \) is smooth and decays at \( \infty \). (Smoothness is automatic, and decay is nearly so.)

2. Note that

\[
-\Delta(\nabla \cdot \psi) = \nabla \cdot (-\Delta \psi) = \nabla \cdot \omega = 0.
\]

The fact that \( \omega \) is smooth with rapid decay implies that \( \nabla \cdot \psi \) is also smooth and decays at \( \infty \). Then Liouville’s Theorem implies that \( \psi = 0 \).

3. Define \( u = \nabla \times \psi \). It is obvious that \( \nabla \cdot u = 0 \), and

\[
\nabla \times u = \nabla \times \nabla \times \psi = -\Delta \psi - \nabla(\nabla \cdot \psi) = \omega.
\]

Thus \( u \) has the desired properties,

Using classical formulas for the solution of the equation \( -\Delta \psi = \omega \), we obtain from the above the explicit formula

\[
\begin{aligned}
u(x) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x - y) \times \omega(y)}{|x - y|^3} \, dy.
\end{aligned}
\]

This formula is called the Biot-Savart Law. It has analogs on suitable bounded domains and in Euclidean spaces of different dimensions, the most important case being \( \mathbb{R}^2 \).

We will write \( K \) to denote the \( 3 \times 3 \) matrix-valued function such that

\[
K(z) = \frac{1}{4\pi |z|^3} \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix}, \quad \text{so that } K(z)v := \frac{z \times v}{4\pi |z|^3} \text{ for all } v \in \mathbb{R}^3,
\]

and we will use the notation

\[
\mathcal{K} \omega = K \star \omega, \quad \text{that is, } \mathcal{K} \omega(x) = \int_{\mathbb{R}^3} K(x - y) \omega(y) \, dy.
\]
2. Quick Introduction to the Euler Equations

We have been a little vague about smoothness and decay, but with the formula in hand one can check “by hand” that $u = K\omega$ solves problem (11).

We can combine the Biot-Savart law with the equation for the evolution of vorticity to obtain a system of equations that (at least formally) is closed:

$$
\begin{align*}
\partial_t \omega + (u \cdot \nabla) \omega &= (\omega \cdot \nabla) u \\
u &= K\omega.
\end{align*}
$$

The system (13) is sometimes called the vorticity-stream formulation of the Euler equations.

2.6. Vorticity in 2 dimensions. Given a solution $u = (u^1, u^2)$ in 2 dimensions, let

$$
\tilde{u}(x_1, x_2, x_3) := (u^1(x_1, x_2), u^2(x_1, x_2), 0).
$$

It is then immediate that $\tilde{u}$ solves the 3-d Euler equations, with vorticity

$$
\tilde{\omega}(x_1, x_2, x_3) = (0, 0, (\partial_1 u^2 - \partial_2 u^1)(x_1, x_2)).
$$

Motivated by this, for the 2-d Euler equations we define the (scalar) vorticity to be

$$
\omega = \partial_1 u^2 - \partial_2 u^1 =: \nabla \perp \cdot u \quad \text{for } \nabla \perp := (-\partial_2, \partial_1).
$$

Reasoning more or less as above, we see that if $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}$ is sufficiently smooth and decays at $\infty$, then

$$
u(x) := \int_{\mathbb{R}^2} \frac{1}{2\pi} \frac{(x - y) \perp}{|x - y|^2} \omega(y) \, dy,
$$

solves

$$
\nabla \cdot u = 0, \quad \nabla \perp \cdot u = \omega.
$$

This is the 2-d version of the Biot-Savart Law. We will use the same notation for the Biot-Savart law in 2 and 3 dimensions, trusting that the meaning will be clear from the context. Thus, for 2d problems we will write

$$
K\omega(x) = K * \omega = \int_{\mathbb{R}^2} K(x - y) \omega(y) \, dy, \quad K(z) = \frac{z \perp}{2\pi|z|^2}.
$$

It is sometimes useful to write the Biot-Savart kernel in the form

$$
K\omega = -\nabla \perp (G * \omega), \quad G(x) = \frac{1}{2\pi} \log\left(\frac{1}{|x|}\right).
$$

Noting that $(\tilde{\omega} \cdot \nabla)\tilde{u} = \omega \partial_3 \tilde{u} = 0$, we see that the vorticity-stream formulation of the Euler equations reduces to

$$
\begin{align*}
\partial_t \omega + u \cdot \nabla \omega &= 0 \\
u &= K\omega.
\end{align*}
$$

One can similarly verify that the 3-d vorticity transport formula reduces in 2 dimensions to

$$
\omega(X(\alpha, t), t) = \omega^0(\alpha).
$$

2.7. Existence and uniqueness?

\footnote{In fact $\psi$ in general behaves like $c \log |x|$ as $|x| \rightarrow \infty$, but $\nabla \cdot \psi$ decays like $|x|^{-1}$, so the reasoning is still okay.}
2.7.1. 2 dimensions. In 2d, there is a very satisfactory theory that will be sufficient for most of our needs.

Before stating it, we introduce some definitions and notation:

- First, if X and Y are Banach spaces (for example, spaces of functions, possibly vector-valued, defined on the same domain D) and f ∈ X ∩ Y, then we will write
  $\|f\|_{X \cap Y} := \|f\|_X + \|f\|_Y$.

It is standard, and very easy to check, that this defines a norm on X ∩ Y, and that with this norm, X ∩ Y is a Banach space.

- Second, let X be a Banach space of $\mathbb{R}^k$-valued functions on D for some $k \geq 1$. For $1 \leq p \leq \infty$, and $T \in (0, \infty]$, we define
  $L^p((0, T); X) := \{f : D \times [0, T) \to \mathbb{R}^k | \|f\|_{L^p((0, T); X)} < \infty\}$

where
  $\|f\|_{L^p((0, T); X)} := \|F\|_{L^p([0, T); X)}$ for $F(t) := \|f(\cdot, t)\|_X$.

It is again a standard fact that this defines a norm that makes $L^p((0, T); X)$ into a Banach space. It is common to abbreviate $L^p((0, T); X)$ by $L^p_X$ for example $L^p_T(X)$ or $L^p_X$ or even just $L^p X$.

Similarly, $C([0, T); X)$ is the space of functions that are continuous from the interval $[0, T)$ into the Banach space X, with the norm
  $\|f\|_{C([0, T); X)} := \sup_{t \in [0, T)} \|f(\cdot, t)\|_X$.

- A function $f$ on a domain $D \subset \mathbb{R}^N$ is said to be log-Lipschitz if
  $\|f\|_{LL} := \sup_{0 < \|x - y\| < 1} \frac{|f(x) - f(y)|}{|x - y| (1 + \log \frac{1}{|x - y|})}$

We will write $LL(D)$ to denote the space of all log-Lipschitz functions on $D$.

We state without proof the following basic result.

**THEOREM 1 (Yudovich’s Theorem).** For any $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, there exists $\omega \in L^\infty([0, \infty); L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ that solves the vorticity-stream formulation of the Euler equations, globally in time, in the (weak) sense sense that

- setting $u(\cdot, t) := \nabla \omega(\cdot, t)$ for every $t$, we have $u(\cdot, t) \in LL(\mathbb{R}^2) \cap W^{1,p}_{\text{loc}}(\mathbb{R}^2)$ for every $p < \infty$, and $\nabla^\perp \cdot u(\cdot, t) = \omega(\cdot, t)$.

- for every $T > 0$ and every bounded $\phi \in C_0^1(\mathbb{R}^2 \times [0, T])$ the identity

$$\int_{\mathbb{R}^2} \omega \phi \, dx \bigg|_0^T = \int_0^T \int_{\mathbb{R}^2} \frac{D\phi}{Dt} \omega \, dx \, dt$$

holds.

Moreover

- The particle trajectory map $\alpha \mapsto X(\alpha, t)$, solving (as usual)
  $\partial_t X(\alpha, t) = u(X(\alpha, t), t)$, \hspace{1cm} $X(\alpha, 0) = \alpha$
  is a well-defined bijection $\mathbb{R}^2 \to \mathbb{R}^2$ that is area-preserving and Hölder continuous.
• **The vorticity transport formula**

\[
\omega(X(\alpha, t), t) = \omega_0(\alpha)
\]

holds.

The notion of weak solution in Theorem 1 is strong enough that it yields expected properties of 2d Euler flows, such as

\[
\|\omega(\cdot, t)\|_{L^p(\mathbb{R}^2)} = \|\omega_0\|_{L^p(\mathbb{R}^2)} \quad \text{for all } t > 0.
\]

Some heuristics behind the theorem are as follows:

- If \( u \) were Lipschitz continuous, then the equation defining the particle trajectory map would be well-posed.
- Elliptic regularity just fails to guarantee that \( u \) is Lipschitz continuous. Indeed, for every \( p \in (1, \infty) \), standard facts (the Calderon-Zygmund estimates) imply that

\[
\|\nabla u\|_{L^p} \leq C_p \|\omega\|_{L^p}.
\]

But this fails for \( p = \infty \).
- However, if \( \omega \in L^\infty \) and \( u = \mathcal{K}\omega \), then \( u \) is log-Lipschitz, and \( \|u\|_{L^\infty} \leq C\|\omega\|_{L^\infty} \).
- And it turns out that basic ODE results, for which Lipschitz continuity is a standard hypothesis, continue to hold if one assumes only log-Lipschitz.

2.7.2. **N ≥ 3 dimensions.** In 3 and higher dimensions, the Euler equations on all of \( \mathbb{R}^N \) are known to have smooth, decaying solutions for smooth, decaying initial data. For example, if \( u_0 \in H^m(\mathbb{R}^N) \) for \( N \geq \left\lfloor \frac{N}{2} \right\rfloor + 2 \), then the Euler equations have a solution that is guaranteed to exist for times \( t \in [0, T) \), where

\[
\frac{1}{C_m\|u_0\|_{H^m}}.
\]

Problems with concentrated vorticity lead to initial data, say \( u_{0\varepsilon} \), depending on a small parameter \( \varepsilon \), with the property that

\[
\|u_{0\varepsilon}\|_{H^m} \geq C\varepsilon^{-m},
\]

so these results are of no use for the problems we will consider. Indeed, the absence of any relevant well-posedness results is an indication of the difficulty of questions about concentrated vortex filaments in 3 dimensions.

2.7.3. **however.....** Certain classes of problems in 3 or higher dimensions admit symmetry reductions that allow them to be written as PDEs for functions of 2 variables. The most important example is problems in 3D with cylindrical symmetry. For such problems, well-posedness results are typically available. We will discuss these as needed.

2.8. **Conserved quantities.**
2.8.1. *for the 3D Euler equations.* Let \( u \) be a smooth solution of the Euler equations on \( \mathbb{R}^3 \), with rapid decay, and let \( \omega = \nabla \times u \). The following quantities are independent of \( t \):

- total velocity flux
  \[ \int_{\mathbb{R}^3} u \, dx. \]
- total vorticity flux
  \[ \int_{\mathbb{R}^3} \omega \, dx. \]
- kinetic energy
  \[ \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \, dx \]
- helicity
  \[ \int_{\mathbb{R}^3} u \cdot \omega \, dx \]

We omit the proofs of the above facts, which can be found in many texts.

Writing the Euler equations in the form
\[ \partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0 \]
as in (7) (to which we refer for the notation) then taking the curl, we find that the vorticity equations can be written
\[ \partial_t \omega + \nabla \times (\nabla \cdot (u \otimes u)) = 0. \]

Multiplying by a vector field \( \phi : \mathbb{R}^3 \to \mathbb{R}^3 \) and integrating by parts twice (justified if we assume sufficient smoothness and decay) we find that
\[ \partial_t \int_{\mathbb{R}^3} \phi \cdot \omega \, dx = \int_{\mathbb{R}^3} \nabla (\nabla \times \phi) : u \otimes u \, dx. \]

In particular \( \int \phi \cdot \omega \) is conserved whenever the \( 3 \times 3 \) matrix \( \nabla (\nabla \times \phi) \) is antisymmetric (always assuming sufficient smoothness and decay). In particular this holds for any linear \( \phi \) and certain quadratic \( \phi \), including
\[
\begin{pmatrix}
  x_1^2 + x_3^2 \\
  0 \\
  0
\end{pmatrix}, \quad \begin{pmatrix}
  0 \\
  x_1^2 + x_3^2 \\
  0
\end{pmatrix}, \quad \begin{pmatrix}
  0 \\
  x_1 x_2 \\
  x_1 x_3
\end{pmatrix}, \quad \begin{pmatrix}
  x_1 x_2 \\
  0 \\
  x_2 x_3
\end{pmatrix}, \quad \begin{pmatrix}
  x_1 x_3 \\
  x_2 x_3 \\
  0
\end{pmatrix}.
\]

Since are quadratic functions of \( x \), rapid decay of the vorticity is needed to justify the above integration by parts. Certain linear combinations of these have been given names:

- fluid impulse
  \[ \frac{1}{2} \int_{\mathbb{R}^3} x \times \omega \, dx \]
- moment of fluid impulse
  \[ \int_{\mathbb{R}^3} x \times x \times \omega \, dx. \]
2. QUICK INTRODUCTION TO THE EULER EQUATIONS

2.8.2. for the 2D Euler equations. Let \( u \) be a smooth solution of the Euler equations on \( \mathbb{R}^2 \), with reasonable decay, and let \( \omega = \nabla \times u \). The following quantities are independent of \( t \).

- total velocity flux and total vorticity flux
  \[ \int_{\mathbb{R}^2} u \, dx , \quad \int_{\mathbb{R}^2} \omega \, dx . \]
- kinetic energy
  \[ \frac{1}{2} \int_{\mathbb{R}^2} |u|^2 \, dx \]
- pseudo-energy
  \[ \frac{1}{2} \int_{\mathbb{R}^2} \omega (G \ast \omega) \, dx, \quad G = -\frac{\log |x|}{2\pi}, \quad G \ast \cong (-\Delta)^{-1} . \]
- helicity = 0.
- first and second moment of the vorticity
  \[ \frac{1}{2} \int_{\mathbb{R}^2} x \perp \omega \, dx, \quad \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \omega \, dx . \]

The following lemma explains the usefulness of the pseudo-energy. Since we just want to illustrate a point, we do not aim for sharp hypotheses.

**Lemma 3.**

1. If \( u \) is sufficiently smooth with rapid decay, then the kinetic energy and the pseudo-energy are equal.

2. If \( \omega \in L^\infty(\mathbb{R}^2) \) with compact support, and if \( \int_{\mathbb{R}^2} \omega \neq 0 \), then the kinetic energy is infinite and the pseudo-energy is finite.

The lemma suggests that the pseudo-energy may be a good substitute for the kinetic energy in situation in which the latter is infinite.

**Proof.**

1. Let \( \psi = G \ast \omega \), so that \( u = -\nabla \perp \psi \) and \( \omega = -\Delta \psi \). Then for any \( R > 0 \),

\[
\frac{1}{2} \int_{B_R} |u|^2 = \frac{1}{2} \int_{B_R} |\nabla \perp \psi|^2 = \frac{1}{2} \int_{B_R} |\nabla \psi|^2 \\
= \frac{1}{2} \int_{B_R} (-\Delta \psi) \psi + \frac{1}{2} \int_{\partial B_R} \psi \nabla \cdot \nabla \psi \\
= \frac{1}{2} \int_{B_R} \omega \psi - \frac{1}{2} \int_{\partial B_R} \psi (u \cdot \tau) 
\]

where \( \tau = \nabla \perp \) is the (counterclockwise) unit tangent to \( \partial B_R \). We now stipulate that “sufficiently smooth with rapid decay” means that \( \omega \psi \in L^1(\mathbb{R}^2) \) and that \( \lim_{R \to \infty} \int_{\partial B_R} \psi (u \cdot \tau) = 0 \). Under these assumptions we may send \( R \to \infty \) to conclude that

\[ \frac{1}{2} \int_{\mathbb{R}^2} |u|^2 = \frac{1}{2} \int_{\mathbb{R}^2} \omega \psi . \]

2. Assume for concreteness that \( \text{supp}(\omega) \subset B_R(0) \) with \( R \geq 1 \), and that \( \int_{\mathbb{R}^2} \omega = \gamma \neq 0 \). We show in Example 2 in Section 3.1 below that under these hypotheses,

\[ |u(x)| = \frac{\gamma}{2\pi |x|} + O(|x|^{-2}) \quad \text{as } |x| \to \infty . \]
Then it is clear that $u \not \in L^2(\mathbb{R}^2)$. On the other hand, for $x, y \in B_R$,
\[ |\log \frac{1}{|x-y|} | \leq \begin{cases} \log \frac{1}{|x-y|} & \text{if } |x-y| \leq 1 \\ |\log 2R| & \text{if } |x-y| \geq 1 \end{cases} \]
It follows that for $x \in B_R$,
\[ |\psi(x)| = \left| \int_{B_R} \omega(y) \frac{1}{2\pi} \log \frac{1}{|x-y|} \, dy \right| \leq \|\omega\|_{L^\infty} 2\pi \left( \int_{|z|<1} \log \frac{1}{|z|} \, dz + \pi R^2 \log 2R \right). \]
Thus
\[ \left| \int_{\mathbb{R}^2} \omega \psi \, dx \right| \leq (\sup_{B_R} |\psi|) \int_{B_R} |\omega| \, dx < \infty. \]

2.9. Some exercises.

1. Assume that $X : D \times [0,T) \to D$ is a $C^2$ particle trajectory map, and let $u$ be the corresponding velocity field, defined by (3). Prove that
\[ \frac{\partial}{\partial t} X(\alpha, t) = (\partial_t u + (u \cdot \nabla) u)_{|x, t, \dot{t}} = (X(\alpha, t), t). \]

2. Prove that if $X$ is a $C^2$ particle trajectory map and $u$ the corresponding velocity field, then the conditions in (4) are equivalent as asserted.

3. Recalling the identities, valid for any smooth enough vector fields $v, w$ in $\mathbb{R}^3$
\[ (v \cdot \nabla) v = \frac{1}{2} \nabla |v|^2 - v \times (\nabla \times v) \]
\[ \nabla \times (v \times w) = -(v \cdot \nabla) w + (w \cdot \nabla) v + (\nabla \cdot v) w - (\nabla \cdot w) v, \]
verify that for smooth solutions, the Euler equations (5) imply the vorticity equation
\[ \partial_t \omega + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u. \]

4. Let $X : D \times [0,T) \to D$ be a $C^2$ particle trajectory map with velocity field $u$. For an open $O \subset D$ and smooth $f : D \times [0,T) \to \mathbb{R}$, prove that
\begin{align*}
\text{transport1} \quad (18) \quad & \frac{d}{dt} \int_{X(O,t)} f \, dx = \int_{X(O,t)} [\partial_t f + \nabla \cdot (uf)] \, dx \\
\text{transport2} \quad (19) \quad & \frac{d}{dt} \int_{X(O,t)} f \, dx = \int_{X(O,t)} \frac{Df}{Dt} \, dx.
\end{align*}

3. Concentrated vorticity: examples and expectations

Our overall goal in these notes is to try to prove theorems about solutions of the Euler equation with initial data for which the vorticity is concentrated.

In this section, we will mostly ignore the $t$ variable, and use the Biot-Savart law to estimate the velocity field generated (at some fixed $t$, for example $t = 0$) by examples of what we have in mind by "concentrated vorticity".

Several examples below use the following well-known lemma about harmonic functions.
Lemma 4. Assume that $\eta \in L^1(\mathbb{R}^N)$ is a.e. radial, that $\text{supp}(\eta) \subset B_R(0)$, and that $h$ is a function (possibly vector- or matrix-valued) that is harmonic on $B_R(0)$. Then

$$\int h(y)\eta(y)\,dy = h(0)\int \eta(y)\,dy.$$ 

Proof. The assumption that $\eta$ is radial means that there exists some $\tilde{\eta} : [0, \infty) \rightarrow \mathbb{R}$ such that $\eta(x) = \tilde{\eta}(|x|)$. Then

$$\int h(y)\eta(y)\,dy = \int_0^R \left( \int_{\partial B_r(0)} h(y)\eta(y)\,d\mathcal{H}^{N-1} \right)\,dr = \int_0^R \tilde{\eta}(r) \left( \int_{\partial B_r(0)} h(y)\,d\mathcal{H}^{N-1} \right)\,dr$$

Here $\mathcal{H}^{N-1}$ denotes $N-1$-dimensional Hausdorff measure; in this setting, the integral is just the standard integration with respect to arclength if $N = 2$, surface area if $N = 3$, and $N-1$-volume for $N \geq 4$. Since $h$ is harmonic on $B_R(0)$, the Mean Value Theorem for harmonic functions implies that for $0 < r \leq R$,

$$\int_{\partial B_r(0)} h(y)\,d\mathcal{H}^{N-1} = h(0)\int_{\partial B_r(0)} \tilde{\eta}(r)\,d\mathcal{H}^{N-1}.$$ 

It follows that

$$\int h(y)\eta(y)\,dy = h(0)\int_0^R \tilde{\eta}(r)\int_{\partial B_r(0)} \eta(y)\,d\mathcal{H}^{N-1}\,dr = h(0)\int_{B(0)} \eta(y)\,dy.$$

We will also use the following fact (often in conjunction with Lemma 4).

Lemma 5. The Biot-Savart kernel

$$K(x) = \frac{x^+}{2\pi|x|^2} = \begin{cases} \frac{(-x_2, x_1)}{2\pi(x_1^2 + x_2^2)} & \text{if } N = 2 \\ \frac{1}{4\pi|x|^3} \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} & \text{if } N = 3 \end{cases}$$

is harmonic away from $x = 0$.

Proof. To verify this without actually computing $\Delta K$, recall that every component of $K$ has the form $\partial_i G$, where $G$ is the Green’s function for the Laplacian

$$G(x) = \begin{cases} \frac{1}{2\pi} \log |x| & \text{if } N = 2 \\ \frac{1}{4\pi}|x|^{-1} & \text{if } N = 3. \end{cases}$$

(In fact we obtained $K$ exactly by differentiating $G$, though our discussion omitted some details.) Then the claim follows from the fact that any derivative of a harmonic function is again harmonic. \hfill \square

\footnote{A vector- or matrix-valued function is said to be harmonic if each of its components is harmonic.}
3.1. 2D examples.

**Example 1.** We first consider an ideal point vortex concentrated at the origin, carrying one unit of vorticity. This may be represented by a Dirac delta-function at the origin, denoted $\delta_0$, defined by

$$\int f(x)\delta_0 = f(0)$$

due to the bounded continuous $f : \mathbb{R}^2 \to \mathbb{R}$. Applying the Biot-Savart law to $\delta_0$, we find that the associated velocity field is

$$u_0(x) = K\delta_0 = \int K(x - y)\delta_0(\text{d}y) = K(x) = \frac{x}{2\pi|x|^2}.

We restate for emphasis:
The velocity field generated by an ideal point vortex is purely rotational and scales like $1/\text{distance}$.

More generally a linear combination of ideal point vortices has the form

$$\omega = \sum_{m=1}^{M} \gamma_m \delta_{p_m}.$$

Here $p_m \in \mathbb{R}^2$ are the vortex locations, and $\gamma_m \in \mathbb{R} \setminus \{0\}$ is the vorticity associated to the $j$th point vortex. The associated velocity field is

$$u(x) = K\left(\sum_{m=1}^{M} \gamma_m \delta_{p_m}\right) = \sum_{m=1}^{M} \gamma_j \frac{(x - p_m)^\perp}{2\pi|x - p_m|^2}.$$

Note that $K(x) = K\delta_0 \notin L^2_{\text{loc}}(\mathbb{R}^2)$. As remarked earlier, there is no known way of making sense of the Euler equations (even allowing some notion of weak solution) for velocity fields not in $L^2_{\text{loc}}$. Thus it simply does not make sense to try to solve the Euler equations for initial data of the form (21). The best one can do is to study the equations for initial vorticity in a space such as $L^1 \cap L^\infty$ and approximating (21), for example satisfying

$$\text{supp}(\omega_\epsilon) \subset \bigcup_{m=1}^{M} B_\epsilon(p_m), \quad \int_{B_\epsilon(p_j)} \omega_\epsilon = \gamma_j, \quad \gamma_j \omega_\epsilon(x) \geq 0 \text{ in } B_\epsilon(p_j)$$

for some positive $\epsilon \ll 1$. We consider the simplest case of this in the next example.

**Example 2.** Assume that $\omega_\epsilon$ satisfies

$$\text{supp}(\omega_\epsilon) \subset B_\epsilon(0), \quad \int_{\mathbb{R}^2} \omega_\epsilon(y) \text{d}y = 1, \quad \omega_\epsilon \geq 0 \text{ everywhere},$$

and let $u_\epsilon := K\omega_\epsilon$. We claim that there exists a constant $C$ such that

$$|u_\epsilon(x) - K(x)| \leq \frac{C\epsilon}{|x|^2} \quad \text{if } |x| \geq 2\epsilon.$$

To see this, note that consider $x$ such that $|x| \geq 2\epsilon$.

$$u_\epsilon(x) - K(x) = \int_{\mathbb{R}^2} K(x - y) \omega_\epsilon(y) \text{d}y - K(x) = \int_{\mathbb{R}^2} [K(x - y) - K(x)] \omega_\epsilon(y) \text{d}y.$$

\footnote{A particularly easy instance of the convolution of a measure and a function, an operation we may recall from basic analysis.}
3. CONCENTRATED VORTICITY: EXAMPLES AND EXPECTATIONS

since \( \int \omega_\epsilon = 1 \). Clearly

\[
K(x - y) - K(x) = \nabla K(x - \theta y) y \quad \text{for some } \theta \in (0, 1).
\]

Since \( K \) is homogeneous of degree \(-1\), it must be that \( \nabla K \) is homogeneous of degree \(-2\), so there exists some \( C_1 > 0 \) such that \( |\nabla K(x - \theta y)| \leq C_1 |x - \theta y|^{-2} \). And since \( |y| \leq \epsilon < 2\epsilon \leq |x| \), it is clear that \( |x - \theta y|^{-2} \leq 4|x|^{-2} \). It follows that

\[
|u_\epsilon(x) - K(x)| \leq \int_{\mathbb{R}^2} \frac{C\epsilon}{|x|^2} \omega_\epsilon(y) \, dy = \frac{4C_1\epsilon}{|x|^2} = \frac{C\epsilon}{|x|^2}
\]

proving (24). Essentially the same argument shows that

\[
\nabla u_\epsilon(x) = \nabla K(x) + O\left(\frac{\epsilon}{|x|^3}\right) \quad \text{uniformly for } |x| > 2\epsilon.
\]

**Example 3.** Let us now consider radial \( \omega \in L^1(\mathbb{R}^2) \). We claim that \( u := \mathcal{X} \omega \) satisfies

\[
(25) \quad \text{there exists } w : [0, \infty) \to \mathbb{R} \text{ such that } u(x) = w(|x|) x^\perp
\]

and

\[
(26) \quad \text{if supp}(\omega) \subset B_\epsilon(0) \text{ then } u(x) = K(x) \int_{B_{2\epsilon}(0)} \omega \, dx \quad \text{when } |x| > R.
\]

These apply in particular to (radial) concentrations of vorticity satisfying (21).

To prove (25) we compute

\[
u(x) \cdot x = \int_{\mathbb{R}^2} x \cdot \frac{(x - y)^\perp}{|x - y|^2} \omega_\epsilon(y) \, dy.
\]

The radial assumption implies that the integrand is odd with respect to the reflection

\[
y = (y \cdot \frac{x}{|x|}) \frac{x}{|x|} + (y \cdot \frac{x^\perp}{|x|}) \frac{x^\perp}{|x|} \quad \mapsto \quad (y \cdot \frac{x}{|x|}) \frac{x}{|x|} - (y \cdot \frac{x^\perp}{|x|}) \frac{x^\perp}{|x|}
\]

through the line that passes through the origin and \( x \), so the integral must vanish. Thus \( u(x, t) \cdot x = 0 \) everywhere. Similarly,

\[
u(x) \cdot x^\perp = \int_{\mathbb{R}^2} \frac{|x|^2}{|x - y|^2} \omega(y) \, dy
\]

and it is easy to see that, due to the radial character of \( \omega \), the integral on the right depends only on \( |x| \). These observations prove (25).

We obtain (26) directly from Lemmas 4 and 5.

We remark that whenever \( \omega \in L^1(\mathbb{R}^2) \) is radial and has enough regularity to make sense of the calculations below \( (W_1^{1,1} \text{ is certainly enough}) \), it follows from (25) that \( \omega \) (viewed as a function of \( (x, t) \) that is independent of \( t \)) satisfies

\[
\partial_t \omega = 0, \quad u \cdot \nabla \omega = 0, \quad \text{and thus } \partial_t \omega + u \cdot \nabla \omega = 0.
\]

That is, \( \omega \) is a stationary solution of the vorticity-stream formulation (13) of the Euler equations. This implies that there is an enormous family of stationary localized 2d vortices.
3.2. 2d expectations. Hand-waving arguments based on the above examples lead us to expect that ideal point vortex dynamics for the 2D Euler equations should look like

\[ \omega(t) = \sum_{m=1}^{M} \gamma_m \delta_{p_m(t)} \]  

where \( \gamma_j \) is independent of \( t \) and

\[ \frac{dp_m}{dt} = \sum_{\ell \neq m} \frac{\gamma_\ell (p_m - p_\ell)^\perp}{2\pi |p_m - p_\ell|^2} \quad \text{for every } m \in \{1, \ldots, M\}. \]  

This will be the case if the velocity fields generated by each point vortex has no impact at all on its own motion. Then we might imagine that each point vortex is simply convected by the sum of the velocity fields generated by its peers. This is what is predicted in (27), (28). (One can give more convincing formal arguments, but in the interests of moving quickly to proofs, we will not bother with that.)

This leads us to ask: given a solution \((p_1(t), \ldots, p_M(t))\) of (28), do there exist solutions \( u_\varepsilon \) of the 2d Euler equations for \( 0 < \varepsilon \ll 1 \) such that the vorticity satisfies

\[ \omega_\varepsilon(\cdot, t) \approx \sum_{m=1}^{M} \gamma_m \delta_{p_m(t)} \]

This question dates back, in some form, to an 1858 paper of Helmholtz.

Clearly any theorem will have to specify a precise sense in which “\( \approx \)" holds. Some possibilities are

\[ \exists r_\varepsilon \to 0 \text{ as } \varepsilon \to 0 \text{ s.t. } \operatorname{supp} \omega_\varepsilon(\cdot, t) \subset \bigcup_{m=1}^{M} B_{r_\varepsilon}(p_m(t)) \text{ and } \int_{B_{r_\varepsilon}(t)} \omega_\varepsilon \, dx \to \gamma_\varepsilon \]

or

\[ \int_{\mathbb{R}^2} f(x) \omega_\varepsilon(x, t) \, dx \to \sum_{m=1}^{M} \gamma_m f(p_m(t)) \quad \text{for all } f \in C_c(\mathbb{R}^2) \]

for every \( t \in [0, T) \). (It is an easy exercise to check that (29) implies (30).)

We remark that the system of ODEs (28) has several conserved quantities, including

- pseudo-energy
  \[ \sum_{m, \ell=1}^{M} \frac{\gamma_m \gamma_\ell}{2\pi} \log \left( \frac{1}{|p_m - p_\ell|} \right) \]
- total vorticity \( \sum_{m=1}^{M} \) is trivially conserved.
- first moment of vorticity
  \[ \sum_{m} \gamma_m p_m^\perp \]
- second moment of vorticity
  \[ \sum_{m} \gamma_m |p_m|^2 \]
We will consider two specific versions of this question:

3.2.1. **rigid motion.** There are certain solutions of \( \text{(28)} \) that evolve by “rigid motion”, that is, in such a way that \( |p_m(t) - p_\ell(t)| \) is independent of \( t \) for all \( m \neq \ell \). Some examples are

- **Translating solutions:** For \( M = 2 \), fix positive numbers \( \gamma, d \). Then
  \[
  \gamma_1 = -\gamma_2 = \gamma, \quad p_1(t) = (-d, vt), \quad p_2 = (d, vt), \quad v = \frac{\gamma}{4\pi d}
  \]
solves \( \text{(28)} \). One may ask:

  **Question:** Given such a solution, for \( 0 < \epsilon \ll 1 \), does there exist a solution of the Euler equations such that
  \[
  \omega_\epsilon(x_1, x_2, t) = \omega_0^\epsilon(x_1, x_2 - v_\epsilon t)
  \]
  with \( v_\epsilon \to v \) as \( \epsilon \to 0 \) and (for example)
  \[
  \text{supp}(\omega_\epsilon^0) \subset \cup_{m=1}^M B_{\epsilon r}(\pm d, 0)
  \]
  \[
  \frac{1}{\epsilon} \int_{B_{\epsilon r}(\pm d, 0)} \omega_\epsilon^0 = \gamma
  \]
  for some sequence \( r_\epsilon \to 0 \)?

  Again, such results are very classical.

- **Rotating solutions:** for \( M \geq 2 \) and \( d, \gamma > 0 \),
  \[
  p_m(t) = R(\alpha t)\left(\frac{d \cos \frac{2\pi m}{M}}{d \sin \frac{2\pi m}{M}}\right), \quad \alpha = \frac{\gamma}{2\pi d} \sum_{m=1}^{M-1} \frac{\sin \theta_m}{2 - 2 \cos \theta_m}, \quad \theta_m := \frac{2\pi m}{M}
  \]
solves \( \text{(28)} \) with \( \gamma_1 = \ldots = \gamma_M = \gamma \), if

  if I have not made a mistake. (Let me know if I have.) We will ask:

  **Question:** Given such a solution, for \( 0 < \epsilon \ll 1 \), does there exist a solution of the Euler equations of the form
  \[
  \omega_\epsilon(x, t) = \omega_0^\epsilon(R(-\alpha_\epsilon t)x)
  \]
  with \( \alpha_\epsilon \to \alpha \) as \( \epsilon \to 0 \) and (for example)
  \[
  \text{supp}(\omega_\epsilon^0) \subset \cup_{m=1}^M B_{\epsilon r}(\pm d, 0)
  \]
  \[
  \frac{1}{\epsilon} \int_{B_{\epsilon r}(\pm d, 0)} \omega_\epsilon^0 = \gamma
  \]
  for some sequence \( r_\epsilon \to 0 \)?

  Again, such results are very classical.

3.2.2. **General motions.** One can also ask: given a solution (in general non-periodic) of \( \text{(28)} \) on an interval \( [0, T] \) such that \( p_m(t) \neq p_\ell(t) \) in \( [0, T] \) for all \( m \neq \ell \), does there exist a solution of the Euler equations such that \( \text{(29)} \) or \( \text{(30)} \) holds? General theorems of this sort date back to the mid 90s, following earlier results from the mid 80s.

3.2.3. **Other questions.** The point vortex ODEs have periodic solutions that do not evolve by a rigid motion. An example
3.3. 3d examples. We now consider velocity fields generated by the kinds of 3d configurations of concentrated vorticity that we will be interested in.

We start with the idealized example of a vortex filament of zero thickness.

EXAMPLE 4. Let \( \Gamma \) be a closed oriented \( C^1 \) curve in \( \mathbb{R}^3 \) of arclength \( L \), parametrized by a function \( \gamma : [0, L] \to \mathbb{R}^3 \) with periodic \( C^1 \) boundary conditions

\[
\gamma(0) = \gamma(L), \quad \gamma'(0) = \gamma'(L)
\]

and such that \( |\gamma'(s)| = 1 \) everywhere. We would like to consider a distribution of vorticity concentrated along \( \Gamma \), pointing in the direction tangent to the curve given by the orientation, and of unit strength. We will refer to this as a ‘perfect vortex filament’ concentrated on \( \Gamma \).

We represent this by a “delta-function concentrated on the curve”. Toward this end, we will write \( \delta_\Gamma \) to be the vector-valued measure defined by

\[
\delta_\Gamma (\phi) := \int_0^L \phi(\gamma(s)) \cdot \gamma'(s) \, ds \quad \text{for } \phi \in C^0_c(\mathbb{R}^3; \mathbb{R}^3).
\]

(We may also use the notation \( \int_{\mathbb{R}^3} \phi \cdot \, d\delta_\Gamma = \int_{\Gamma} \phi \cdot \tau \), where \( \tau \) denotes the unit tangent given by the orientation.)

By applying the Biot-Savart law\(^6\), we find that the velocity field generated by \( \delta_\Gamma \) is

\[
\mathbf{u}_{\text{from curve}}(x) := \mathcal{K}\delta_\Gamma(x) = \int_{\mathbb{R}^3} \frac{(x - y) \times \delta_\Gamma(dy)}{4\pi|x - y|^3} = \int_0^L \frac{(x - \gamma(s)) \times \gamma'(s)}{4\pi|x - \gamma(s)|^3} \, ds.
\]

We will later study in detail the behaviour of \( \mathbf{u}_{\Gamma} \) at points near \( \Gamma \). For example, assume that \( x \) is a point of the form \( x = \gamma(s_0) + \sigma \nu \), where \( \nu \cdot \gamma'(s_0) = 0 \). We may assume after a reparametrization that

\[
\text{main point: as with 2d, } \mathbf{u}_{\Gamma} \notin L^2_{\text{loc}}, \text{ so there is not hope of directly solving Euler for such data with such singular vorticity. Details likely to appear sometime.}
\]

Another important remark, which follows directly from Lemma\(^5\), is that

\[
\mathbf{u}_{\Gamma} \text{ is smooth and harmonic on } \mathbb{R}^3 \setminus \Gamma.
\]

EXAMPLE 5. A nearly-perfect vortex filament of finite thickness \( \varepsilon \ll 1 \) can be obtained by regularizing a perfect vortex filament. To that end, let \( \Gamma \) and \( \delta_\Gamma \) be as above. For \( \varepsilon \in (0, 1] \), let \( \eta_\varepsilon \in C^\infty_\text{c}(\mathbb{R}^3) \) be a function such that

\[
\text{supp}(\eta_\varepsilon) \subset B_\varepsilon(0), \quad \int_{\mathbb{R}^2} \eta_\varepsilon(y) \, dy = 1, \quad \eta_\varepsilon \geq 0 \text{ everywhere}, \quad \eta_\varepsilon \text{ is radial}.
\]

We will also tacitly assume that

\[
|D^k \eta_\varepsilon(x)| \leq C_k \varepsilon^{-k} \text{ for all } x,
\]

where \( D^k \eta \) represents the collection of all \( k \)th order partial derivatives of \( \eta \). This implies that the support of \( \eta_\varepsilon \) is not concentrated in any ball much smaller than \( B_\varepsilon(0) \).

\(^5\)Strictly speaking we should check that \( \nabla \cdot \delta_\Gamma = 0 \) to know that the Biot-Savart law holds.
We will say that a nearly-perfect vortex filament of thickness $\approx \epsilon$ is one corresponding to the smooth vorticity distribution

$$\omega_{\Gamma,\epsilon}(x) = (\eta_{\epsilon} \ast \delta_{\Gamma})(x) = \int_{\mathbb{R}^3} \eta_{\epsilon}(x - y)\delta_{\Gamma}(dy) = \int_{0}^{L} \eta_{\epsilon}(x - y)\gamma'(s) \, ds.$$ 

Clearly if $\omega_{\Gamma,\epsilon}(x) = 0$ if $\text{dist}(x, \Gamma) = 0$, so $\omega_{\Gamma,\epsilon}$ is supported in a tube of radius $\epsilon$ around $\Gamma$.

The velocity field generated by $\omega_{\Gamma,\epsilon}$ is

$$u_{\Gamma,\epsilon}(x) = K \ast \eta_{\epsilon} \ast \delta_{\Gamma} = \eta_{\epsilon} \ast K \ast \delta_{\Gamma} = \eta_{\epsilon} \ast u_{\Gamma} = \int_{B_\epsilon} \eta_{\epsilon}(y)u_{\Gamma}(x - y) \, dy,$$

since the operation of convolution is commutative. As we have assumed that $\eta_{\epsilon}$ is radial, it now follows directly from (33) and Lemma 4 that

$$u_{\Gamma,\epsilon}(x) = u_{\Gamma}(x) \text{ if } \text{dist}(x, \Gamma) \geq \epsilon.$$

### 3.4. 3d conjectures.

An ideal version of these notes would give more detailed computations of the behaviour of $u_{\Gamma}$ near $\Gamma$ above, and would use them here to motivate conjectures on how actual solutions behave. These issues will be discussed later.

---

### 4. An extremely quick introduction to the Gross-Pitaevskii equation

#### 4.1. Some basics.

We recall that the Gross-Pitaevskii equation

$$i\partial_t \psi - \Delta \psi + \frac{1}{\epsilon^2}(|\psi|^2 - 1)\psi = 0, \quad 0 < \epsilon \ll 1$$

for $\psi : D \times [0, T) \to \mathbb{C}$, where $D$ is an open subset of $\mathbb{R}^N$, often the entire space.

One may think of the limit $\epsilon \to 0$ as the incompressible limit; in this regime. If $\partial D \neq \emptyset$ then we will normally impose Neumann boundary conditions

$$\nu \cdot \nabla \psi = 0 \quad \text{on } \partial D.$$

**Notation:** For $v, w \in \mathbb{C}$, we will write

$$\langle v, w \rangle := \Re(v\bar{w}) = \frac{1}{2}(v\bar{w} + \bar{v}w).$$

Given $v \in \mathbb{C}$, we will sometimes write $v_1 = \Re(v)$ and $v_2 = \Im(v)$. Note that

$$\langle iv, w \rangle = \langle w, iv \rangle = -\langle iv, w \rangle = -\langle v, iw \rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}.$$

A solution $\psi$ is interpreted as a wave function describing an ideal dilute quantum mechanical fluid. Physical quantities encoded in the wave function include the following:

- **Energy density**

$$e_{\epsilon}(\psi) := \frac{1}{2} |\nabla \psi|^2 + \frac{(|\psi|^2 - 1)^2}{4\epsilon^2}.$$

- **Mass density**

$$|\psi|^2.$$
• current, or momentum density
\[ j(\psi) := (i\psi, \nabla \psi) = \text{vector with kth component} \ (i\psi, \partial_k \psi). \]
If one writes \( \psi = \rho e^{i\phi} \), then \( j(\psi) = \rho^2 \nabla \phi. \)

• The vorticity is defined by
\[ \omega(\psi) = \begin{cases} 
\frac{1}{2} \nabla \times j(\psi) & \text{if } N = 3 \\
\frac{1}{2} \nabla^\perp \cdot j(\psi) & \text{if } N = 2.
\end{cases} \]

A short computation using (38) shows that for \( N = 3 \)
\[ \omega(\psi) = \nabla \psi_1 \times \nabla \psi_2. \]

(This is the reason the normalization of \( \frac{1}{2} \) in the definition of \( \omega \) is natural.) As with the Euler equations, we can define a vortex line to be an integral curve of \( \omega(\psi) \). It follows from (39) that \( \omega(\psi) \) is orthogonal to \( \nabla \psi_k \) for \( k = 1, 2 \) and hence parallel to level sets of both \( \psi_1 \) and \( \psi_2 \). Thus if \( \psi \) is \( C^1 \) say, then \( \omega(\psi) \) is parallel to level curves of the complex function \( \psi \). It follows that

nondegenerate level curves of \( \psi \) are vortex lines

where we say that a level curve is nondegenerate if \( \omega(\psi) \) does not vanish on it.

The quantities introduced above all satisfy conservation laws. We will write these in differential form. We will omit the easy verifications of the first two:

\[ \frac{d}{dt} \frac{\partial}{\partial t} e_\epsilon (\psi) = \nabla \cdot (\partial_t \psi, \nabla \psi). \]  
\[ \frac{d}{dt} |\psi|^2 \]  
\[ = \nabla \cdot j(\psi). \]

Below we write \( \nabla \psi \otimes \nabla \psi \) to indicate the \( N \times N \) matrix whose \((k, \ell)\) entry is \( \partial_k \psi \partial_\ell \psi \). Then

\[ \frac{d}{dt} j(\psi) = 2 \nabla \cdot (\nabla \psi \otimes \nabla \psi) - \nabla [2e_\epsilon (\psi) + (i\psi_t, \psi)] \]

and if \( N = 3 \) then
\[ \frac{d}{dt} \omega(\psi) = \nabla \times (\nabla \cdot (\nabla \psi \otimes \nabla \psi)) \]

\textbf{Proof of (42) and (43).} First,
\[ (i\psi_t, \psi_{x_k}) = \partial_k (i\psi_t, \psi) - (i\psi_{tx_k}, \psi) \quad \text{and} \]
\[ (i\psi_t, \psi_{x_k}) = \partial_t (i\psi, \psi_{x_k}) - (i\psi_t, \psi_{tx_k}). \]

If we add these, the second derivative terms cancel (recalling (38)) and we obtain
\[ \partial_t (i\psi_t, \psi_{x_k}) = -\partial_k (i\psi_t, \psi) + 2(i\psi_t, \psi_{x_k}). \]

Next, using (35) we have
\[ (i\psi_t, \psi_{x_k}) = (\Delta \psi, \psi_{x_k}) - \partial_{x_k} \left( \frac{(|\psi|^2 - 1)^2}{4\epsilon^2} \right), \]

and
\[ (\Delta \psi, \psi_{x_k}) = (\psi_{x_k} x_t, \psi_{x_k}) = (\psi_{x_t}, \psi_{x_k}) x_t - (\psi_{tx_k}, \psi_{x_t}) \]
\[ = (\psi_{x_t}, \psi_{x_k}) x_t - \frac{1}{2} |\nabla \psi|^2_{x_k}. \]
4. AN EXTREMELY QUICK INTRODUCTION TO THE GROSS-PITAEVSKII EQUATION

We obtain (42) by assembling these computations, and (43) follows by taking the curl of (42).

4.2. formal similarities between Euler and Gross-Pitaevskii. Let \( \psi_\varepsilon \) be to denote a solution of (35) for some value of \( \varepsilon \ll 1 \).

The basic analogy between the Euler and the Gross-Pitaevskii equations is that for the problems we will consider

\[
\begin{align*}
\text{j}(\psi_\varepsilon) & \text{ is analogous to the velocity u in Euler, and thus} \\
\text{ω}(\psi_\varepsilon) & \text{ is analogous to the vorticity ω in Euler.}
\end{align*}
\]

We attempt to give some crude arguments in favor of these claims.

To start, note that it follows easily from the differential conservation law (40) and the boundary conditions (36) that for a solution of (35),

\[
E_\varepsilon(\psi_\varepsilon(\cdot, t)) = \int_D e_\varepsilon(\psi_\varepsilon(x, t)) \, dx
\]

is independent of \( t \).

The energy scaling for the initial data that we consider will turn out to be

\[
E_\varepsilon(\psi_\varepsilon^0) \approx |\log \varepsilon|.
\]

Under these hypotheses, the solution to (35) will satisfy

\[
\int_D \frac{(\psi_\varepsilon(x, t))^2 - 1}{4\varepsilon^2} \, dx \leq E_\varepsilon(\psi_\varepsilon(\cdot, t)) \leq C|\log \varepsilon|
\]

and thus

\[
\| |\psi(\cdot, t)|^2 - 1\|_{L^2} \leq C\varepsilon \sqrt{|\log \varepsilon|}.
\]

It follows that in this scaling, \( \text{j}(\psi_\varepsilon) \) is approximately divergence-free for \( 0 < \varepsilon \ll 1 \), in the sense that (from (41))

\[
\nabla \cdot j(\psi_\varepsilon) = \frac{1}{2} \frac{d}{dt}(|\psi|^2 - 1) \approx 0
\]

where “\( \approx 0 \)" means “of size \( \approx \varepsilon|\log \varepsilon|^{1/2} \) in a very weak norm." Weak though the norm is, this provides a sense in which \( j(\psi_\varepsilon) \) approximately shares the key divergence-free property of the velocity \( u \) in Euler.

Second, we argue that the equation satisfied by the vorticity \( \omega(\psi_\varepsilon) \), which we recall here,

\[
\frac{d}{dt} \omega(\psi_\varepsilon) = \nabla \times (\nabla \cdot (\nabla \psi_\varepsilon \otimes \nabla \psi_\varepsilon))
\]

is similar, for \( E_\varepsilon(\psi_\varepsilon) \approx |\log \varepsilon| \), to the vorticity equation for Euler. In support of this, we write (where \( \psi \neq 0 \))

\[
\partial_k \psi_\varepsilon = (\partial_k \psi_\varepsilon, \frac{\psi_\varepsilon}{|\psi_\varepsilon|}) \left[ \frac{\psi_\varepsilon}{|\psi_\varepsilon|} \right] + (\partial_k \psi_\varepsilon, \frac{i\psi_\varepsilon}{|\psi_\varepsilon|}) \left[ \frac{i\psi_\varepsilon}{|\psi_\varepsilon|} \right] = \partial_k |\psi_\varepsilon| \frac{\psi_\varepsilon}{|\psi_\varepsilon|} + j_k(\psi_\varepsilon) \frac{i\psi_\varepsilon}{|\psi_\varepsilon|}.
\]

Since \( \left( \frac{\psi_\varepsilon}{|\psi_\varepsilon|}, \frac{i\psi_\varepsilon}{|\psi_\varepsilon|} \right) \) form an orthonormal basis for \( \mathbb{R}^2 \), it follows that

\[
(\partial_k \psi_\varepsilon, \partial_\ell \psi_\varepsilon) = (\partial_k |\psi_\varepsilon|, \partial_\ell |\psi_\varepsilon|) + \frac{j_k(\psi_\varepsilon) \cdot j_\ell(\psi_\varepsilon)}{|\psi|^2}.
\]

In tensor notation,

\[
\nabla \psi_\varepsilon \otimes \nabla \psi_\varepsilon = \nabla |\psi_\varepsilon| \otimes \nabla |\psi_\varepsilon| + \frac{j(\psi_\varepsilon) \otimes j(\psi_\varepsilon)}{|\psi_\varepsilon|^2}.
\]
When $0 < \varepsilon \ll 1$ and $E_\varepsilon(\psi) \approx |\log \varepsilon|$, as we have seen, $|\psi_\varepsilon| \approx 1$ and thus we may imagine that $\nabla|\psi_\varepsilon| \approx 0$. If we believe these, then we may conclude that

$$\nabla\psi_\varepsilon \otimes \nabla\psi_\varepsilon \approx j(\psi_\varepsilon) \otimes j(\psi_\varepsilon).$$

Substituting this into the equation for the vorticity, we finally obtain

$$\partial_t \omega(\psi_\varepsilon) \approx \nabla \times (\nabla \cdot j(\psi_\varepsilon) \otimes j(\psi_\varepsilon)).$$

And if we believe in the analogies (44) then this looks a lot like the vorticity equation for Euler:

$$\partial_t \omega = -\nabla \times (\nabla \cdot (u \otimes u))$$

(up to a sign and some scaling factors, e.g. the factor of $1/2$ in the definition $\omega(\psi) = \frac{1}{2} \nabla \times j(\psi)$.)

As mathematicians, we must concede that this argument is so far from rigorous as to be totally unbelievable. But as we will see, there are many similarities between the behavior of concentrated vortices in Euler and Gross-Pitaevskii, and this non-rigorous argument in fact provides the basis for proofs.

4.3. Additional facts about Gross-Pitaevskii. Here are some of the many important differences between Gross-Pitaevskii and Euler.

- **There is no Biot-Savart Law for Gross-Pitaevskii.**
- **There is no vorticity transport formula for Gross-Pitaevskii.**
- **In 2 dimensions, there is no very natural pseudo-energy for Gross-Pitaevskii.**

The above all make it harder to prove theorems for GP than for Euler, since many tools are available for the latter but not the former.

On the other hand, there are also a number of ways in which GP is easier than Euler.

- **In GP, the length scale $\varepsilon$ is built into the equation and into the conserved energy.** This will have numerous useful consequences.
- **The structure of 2d stationary vortices is much more rigid for GP than for Euler.** We have already seen a lack of rigidity in Euler, where any radial distribution of vorticity is stationary with respect to the time evolution. We may see later on what the greater rigidity for GP means in practice.

We also discussed some other things, not yet typed. Anyway, when we begin to prove theorems, we will start almost from scratch.
CHAPTER 2

Point vortices in the Euler equations in 2 dimensions

Our goal in this chapter is to prove theorems establishing the existence of solutions of the Euler equations in $\mathbb{R}^2$ for which $\omega_\varepsilon \rightharpoonup \sum_{m=1}^M \gamma_m \delta_{p_m(t)}$ for $p(y) = (p_1(t), \ldots, p_M(t))$ solving

$$\frac{dp_m}{dt} = \sum_{\ell \neq m} \frac{\gamma_{\ell} (p_m - p_{\ell})^\perp}{2\pi |p_m - p_{\ell}|^2} \quad \text{for every } m \in \{1, \ldots, M\}.$$  

1. rigid motions

We have seen that the point vortex ODE system (45) (with $\gamma_1 = \ldots = \gamma_m = \gamma$) has rigidly rotating solutions, that is, solutions of the form

$$p_m(t) = R(\alpha t) \left( \frac{d \cos \frac{2\pi m}{M}}{d \sin \frac{2\pi m}{M}} \right), \quad \text{for } R(\theta) := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

for $m = 0, \ldots, M - 1$, where $\alpha$ depending on $d, \gamma, M$. At every $t$, the solution has $M$ points evenly spaced around a circle of radius $d$, centred at the origin. We will prove the existence of solutions of the Euler equations for which the vorticity is rigidly rotating, and that approximate a solution (46) of (45).

This is a typical example of a theorem constructing solutions of the Euler equations that evolve by a rigid motion – rotation or translation – and that are “close to” a corresponding solution of the (45).

As we will see, such problems can be approached by either PDE or variational arguments. The proof we give here will be an example of a variational approach; later on we will consider a PDE approach to a related problem.

For simplicity we will set $d = \gamma = 1$; the general case can easily be obtained by scaling and a change of variables. We will prove

**Theorem 2.** For any $M \geq 2$, there exists a sequence of functions $\omega_\varepsilon : \mathbb{R}^2 \to [0, \infty)$ and numbers $\alpha_\varepsilon$ such that

$$\omega_\varepsilon(R(\frac{2\pi}{M})x) = \omega_\varepsilon(x) \quad \text{for a.e. } x \in \mathbb{R}^2$$

$$\omega_\varepsilon(x) \rightharpoonup \sum_{m=1}^M \delta_{(\cos \theta_m, \sin \theta_m)}, \quad \theta_m = \frac{2\pi m}{M}$$
in the sense that
\[ \int_{\mathbb{R}^2} f(x) \omega_\varepsilon(x) \, dx \to \sum_{m=1}^{M} f(\cos \theta_m, \sin \theta_m) \quad \text{for every bounded, continuous } f : \mathbb{R}^2 \to \mathbb{R}, \]
and
\[ \bar{\omega}_\varepsilon(x, t) := \omega_\varepsilon(R(-\alpha_\varepsilon t)x) \]
solves the vorticity-stream formulation of the Euler equations.

Finally, \( \alpha_\varepsilon \to \alpha = \ldots \) as \( \varepsilon \to 0 \).

The proof will yield more information about \( \omega_\varepsilon \) than we have stated here. There are a number of results of this sort. Here are some reasons that we have chosen to present this particular one

- The proof is one of the less technical ones of its type, so it illustrates key ideas and techniques in a relatively simple form
- Some of the ideas that it introduces will be useful for other problems we will consider later, including dynamical questions.
- The result is connected to interesting open questions.

The proof we present follows the paper *Corotating steady vortex flows with N-fold symmetry*. Nonlinear Anal. 9 (1985), no. 4, 351–369, by Bruce Turkington.

Recall that any radial function is a stationary solution of the Euler equations. This suggests that one might expect massive non-uniqueness in the problem considered in Theorem 2. This suspicion is further reinforced by Proposition 1 below. We will therefore look for \( \omega_\varepsilon \) of a particular form. A convenient choice is

\[ \omega_\varepsilon = \sum_{m=1}^{M} \frac{1}{\pi \varepsilon^2} 1_{R(\theta_m)A_\varepsilon} \quad \text{for some measurable } A_\varepsilon \subset \mathbb{R}^2, \]

where for any \( A \subset \mathbb{R}^2 \),
\[ R(\theta)A := \{R(\theta)x : x \in A\} \quad \text{and} \quad 1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if not.} \end{cases} \]

Given \( (50) \), in order for \( (48) \) to hold, it will have to be the case that \( L^2(A_\varepsilon) = \pi \varepsilon^2 \), among other properties, where \( L^2 \) denotes 2-dimensional Lebesgue measure. (I have included the factor of \( \pi \) because I would like to imagine that \( A_\varepsilon \) is roughly a ball of radius \( \varepsilon \).)

Our overall strategy for proving the theorem is as follows.

- We first recast the problem as a variational problem – the problem of maximizing the pseudo-energy subject to some constraints (depending on \( \varepsilon \) and consistent with \( (50) \) and \( (48) \)).
- We then prove the existence of a maximizer \( \omega_\varepsilon \) for small enough \( \varepsilon \).
- Finally, we verify that the function obtained in this way has the form \( (50) \) and satisfies the conclusions of Theorem 2.

1.1. formulation of the variational problem. Since we are working in the symmetry class \( (47) \), it suffices to define \( \omega_\varepsilon \) in the sector

\[ S_0 := \{(r \cos \theta, r \sin \theta) : r \in (0, \infty), |\theta| < \pi/M\}. \]
We will identify functions in $L^1(S_0)$ with functions in $L^1(\mathbb{R}^2)$ that vanish a.e. outside of $S_0$. With this convention, given $\omega \in L^1(S_0)$ we will write

$$\tilde{\omega}(x) := \sum_{m=1}^{M} \omega(R(\theta_m)x).$$

(51)

Clearly, $\tilde{\omega}$ is the unique element of $L^1(\mathbb{R}^2)$ such that

$$\tilde{\omega}(x) = \omega(x) \quad \text{for a.e. } x \in S_0,$$

and $\tilde{\omega}$ has the symmetry (57).

For $\eta: \mathbb{R}^2 \to \mathbb{R}$ we denote the pseudo-energy by

$$E[\eta] := \frac{1}{2} \int_{\mathbb{R}^2} \eta(G * \eta) = \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} G(x-x')\omega(x)\omega(x')dx'dx', \quad G = -\log|x|\frac{1}{2\pi}.$$

We will check below that $E$ is well-defined and finite in the situations that we will consider. For $\omega: S_0 \to \mathbb{R}$ we define

$$\tilde{E}[\omega] := \frac{1}{M} E[\tilde{\omega}].$$

We will seek $\omega_\varepsilon$ supported in

$$S_1 := \left\{ (r\cos \theta, r\sin \theta) : \frac{1}{2} < r < 2, |\theta| < \pi/2M \right\}.$$

(52)

and, as with $S_0$, elements of $L^1(S_1)$ are taken to equal 0 a.e. in $\mathbb{R}^2 \setminus S_1$. We define

$$\Sigma_\varepsilon := \left\{ \omega \in L^1(S_1) : \int_{S_0} \omega \, dx = \int_{S_0} |x|^2 \omega \, dx = 1, \quad 0 \leq \omega \leq \frac{1}{\pi\varepsilon^2} \text{ a.e.} \right\}.$$

The variational problem that we will consider is

$$\text{VP1} \quad \text{Find } \omega_\varepsilon \text{ maximizing } \tilde{E} \text{ in } \Sigma_\varepsilon, \text{ and with compact support in } S_1.$$  

(53)

The proof of Theorem 2 consists of two main parts.

First, we will prove that solutions of problem (53) satisfy the conclusions of Theorem 2.

Second, we will establish the existence of solutions of problem (53). The requirement of compact support in $S_1$ is an unusual feature of the problem and will require some attention.

Before getting started we record some preliminary computations.

First, for $\omega \in L^1(S_0)$ it follows from (51) and a change of variables that

$$G * \tilde{\omega}(x) = \int_{\mathbb{R}^2} G(x-x') \sum_{m=1}^{M} \omega(R(\theta_m)x')dx'$$

$$= \int_{S_0} \sum_{m=1}^{M} G(x-R(\theta_m)x')\omega(x')dx'$$

$$= \int_{S_0} G_M(x,x')\omega(x')dx' \quad \text{for } G_M(x,x') := \sum_{m=1}^{M} G(x-R(\theta_m)x').$$

(54)
Also, it is rather clear that \( G \ast \bar{\omega} \) has the same \( M \)-fold symmetry as \( \bar{\omega} \). Thus \( \bar{\omega}(G \ast \bar{\omega}) \) has the same symmetry. It follows that

\[
\hat{E}(\omega) = \frac{1}{M} \mathcal{E}(\bar{\omega}) = \frac{1}{2M} \int_{\mathbb{R}^2} \bar{\omega}(x)(G \ast \bar{\omega})(x) \, dx
\]

\[
= \frac{1}{2} \int_{S_0} \omega(x)(G \ast \bar{\omega})(x) \, dx
\]

\[
= \frac{1}{2} \int_{S_0 \times S_0} G_M(x, x') \omega(x)\omega(x') \, dx' \, dx
\]

(55)

**1.2. characterization of maximizers.** In this section we prove that solutions of the maximization problem \((\ref{maximizers})\) in fact yield rigidly rotating \( M \)-vortex solutions of Euler.

In fact, we will prove a more general and interesting result. To state it we need a definition: If \( \mathcal{D} \) is an open subset of \( \mathbb{R}^N \) and \( F_1, F_2: \mathcal{D} \to [0, \infty) \) are integrable, we say that \( F_2 \) is a **rearrangement of** \( F_1 \) if

\[
\forall \lambda > 0, \quad \mathcal{L}^N(\{x \in \mathcal{D} : F_1(x) \geq \lambda \}) = \mathcal{L}^N(\{x \in \mathcal{D} : F_2(x) \geq \lambda \}).
\]

(56) **rearrangement.def**

Here are some standard consequence of the definition.

- if \( F_2 \) is a rearrangement of \( F_1 \), then \( \|F_1\|_{L^q} = \|F_2\|_{L^q} \) for all \( q \in [1, \infty] \) (where both sides may equal \( +\infty \) for some values of \( q \)).
- if \( x \mapsto Y(x) \) is a volume-preserving diffeomorphism and \( F \in L^1 \) is a non-negative function, then \( F \circ Y \) is a rearrangement of \( F \)

**prop:benjamin**

**PROPOSITION 1.** Let \( \eta \in L^\infty(S_0) \) be a nonnegative function, and assume that \( \omega_\varepsilon \) maximizes \( \hat{E} \) in the set

\[
\{ \omega \in \mathbb{L}^1(S_0) : \text{ \omega is a rearrangement of } \eta, \text{ \supp}(\omega) \subset \bar{S}_1, \int |x|^2 \omega(x) \, dx = 1 \}.
\]

Further assume that

\[
\omega_\varepsilon \text{ has compact support in } S_1.
\]

(57) **oep.cs**

Let \( \tilde{\omega}_\varepsilon \) denote the \( M \)-fold symmetric extension of \( \omega_\varepsilon \) to \( \mathbb{R}^2 \), as in \((\ref{extension})\). Then

\[
\tilde{\omega}_\varepsilon(x, t) := \tilde{\omega}_\varepsilon(R(-\alpha_\varepsilon t)x)
\]

is a weak (Yudovich) solution of the Euler equations.

We will break the proof into two lemmas. The first is just a change of variables that reduces the Euler equations for \( \bar{\omega} \) on \( \mathbb{R}^2 \times \mathbb{R} \) to a PDE for \( \tilde{\omega} \) on \( \mathbb{R}^2 \).

**LEMMA 6.** Assume that \( \tilde{\omega}_\varepsilon(x, t) \) has the form \( \tilde{\omega}_\varepsilon(x, t) := \tilde{\omega}_\varepsilon(R(-\alpha_\varepsilon t)x) \). Then \( \tilde{\omega}_\varepsilon \) is a weak (Yudovich) solution of the Euler equations if \( \tilde{\omega}_\varepsilon \) satisfies

\[
\int_{\mathbb{R}^2} \tilde{\omega}_\varepsilon(y) \nabla^\perp (\tilde{\psi}_\varepsilon(y) + \frac{\alpha_\varepsilon}{2} |y|^2) \cdot \nabla \tilde{\phi}(y) \, dy = 0 \quad \text{for } \tilde{\phi} \in C^1_c(\mathbb{R}^2),
\]

(59) **EL.corot**

for \( \tilde{\psi}_\varepsilon = G \ast \tilde{\omega}_\varepsilon \).

**REMARK 2.** Note that \((\ref{EL.corot})\) is the weak form of the equation

\[
- \nabla^\perp (\tilde{\psi}_\varepsilon(x) + \frac{\alpha_\varepsilon}{2} |x|^2) \cdot \nabla \tilde{\omega}_\varepsilon = 0, \quad \tilde{\psi}_\varepsilon = (-\Delta)^{-\frac{1}{2}} \tilde{\omega}_\varepsilon.
\]

(60) **EL.corot2**

Here \( \tilde{u}_\varepsilon = -\nabla^\perp \tilde{\psi}_\varepsilon \) is the velocity field generated by the vortices in the rotating frame with respect to which they appear stationary, and \(-\alpha_\varepsilon \nabla^\perp \) is the velocity field of
the rotating frame with respect to the nonrotating frame. So $-\nabla^\perp(\hat{\psi}_\varepsilon(x) + \frac{\alpha\varepsilon}{2}|x|^2)$ is exactly the velocity field associated to the vortices, as seen in the inertial frame.

Note also, if $v, w$ are nonzero vectors in $\mathbb{R}^2$, then $v^\perp \cdot w = 0$ if and only if $v$ and $w$ are parallel, or (equivalently) $v^\perp$ and $w^\perp$ are parallel. Moreover, $\nabla^\perp f$ (when nonzero) is tangent to level curves of $f$. Thus (60) states that

level curves of $\hat{\psi}_\varepsilon(x) + \frac{\alpha\varepsilon}{2}|x|^2$ and $\hat{\omega}_\varepsilon$ have the same tangents, assuming sufficient smoothness. One way to construct functions with this property is to require that $\hat{\omega}_\varepsilon = f(\hat{\psi}_\varepsilon(x) + \frac{\alpha\varepsilon}{2}|x|^2)$ for some $f : \mathbb{R} \to \mathbb{R}$. Thus, since $-\Delta \hat{\psi}_\varepsilon = \hat{\omega}_\varepsilon$, one way to find solutions of (60) (and hence (59)) is to find solutions of the PDE

$$-\Delta \hat{\psi}_\varepsilon = f(\hat{\psi}_\varepsilon + \frac{\alpha\varepsilon}{2}|x|^2).$$

We will not do this here, but later on we will discuss a PDE approach to some related questions.

**Proof.** Fix $T > 0$ and $\hat{\Phi} \in C^1_c(\mathbb{R}^2 \times [0, T])$, and consider the integral

$$\int_0^T \int_{\mathbb{R}^2} \frac{D\hat{\Phi}}{Dt} \hat{\omega} \, dx \, dt = \int_0^T \int_{\mathbb{R}^2} (\partial_t \hat{\Phi}(x, t) + \hat{u}_\varepsilon(x, t) \cdot \nabla \hat{\Phi}(x, t)) \hat{\omega}_\varepsilon(x, t) \, dx \, dt$$

where

$$\hat{u}_\varepsilon(\cdot, t) = \mathcal{K} \hat{\omega}_\varepsilon(\cdot, t).$$

For every $t$ we may change variables, setting $y = R(-\alpha t)x$. We also define $\Phi$ by setting

$$\hat{\Phi}(x, t) = \Phi(R(-\alpha t)x, t) = \Phi(y, t).$$

With this notation, the change of variables leads to

$$\int_0^T \int_{\mathbb{R}^2} \frac{D\hat{\Phi}}{Dt} \hat{\omega} \, dx \, dt = \int_0^T \int_{\mathbb{R}^2} (\partial_t \Phi(y, t) + (-\alpha \varepsilon y^\perp + u_\varepsilon(y)) \cdot \nabla y \Phi(y, t)) \hat{\omega}_\varepsilon(y) \, dy \, dt.$$ 

See the end of the proof for details. Since $-\alpha \varepsilon y^\perp + u_\varepsilon(y) = -\nabla^\perp(\hat{\psi}_\varepsilon(y) + \frac{\alpha\varepsilon}{2}|y|^2)$, the hypothesis (59) implies that most of the terms on the right-hand side integrate to 0, and the above identity reduces to

$$\int_0^T \int_{\mathbb{R}^2} \frac{D\hat{\Phi}}{Dt} \hat{\omega} \, dx \, dt = \int_0^T \int_{\mathbb{R}^2} \partial_t \Phi(y, t) \hat{\omega}(y) \, dy \, dt.$$ 

Since $\Phi$ has compact support, it is clear that $\int \partial_t \Phi(y, t) \hat{\omega}_\varepsilon(y) \, dy = 0$, so the above can be rewritten as

$$\int -\nabla_y^\perp(\hat{\psi}_\varepsilon(y) + \frac{\alpha\varepsilon}{2}|y|^2) \cdot \nabla y \Phi(y, t)) \hat{\omega}_\varepsilon(y) \, dy \, dt.$$ 

Integrating with respect to $t$ and changing variables again, we find that

$$\int_0^T \int_{\mathbb{R}^2} \partial_t \Phi(y, t) \hat{\omega}(y) \, dy \, dt = \int_{\mathbb{R}^2} \Phi(y, t) \hat{\omega}(y) \, dy \bigg|_{t=0}^{t=T} = \int_{\mathbb{R}^2} \hat{\Phi}(x, t) \hat{\omega}(x, t) \, dx \bigg|_{t=0}^{t=T}$$

Thus

$$\int_{\mathbb{R}^2} \hat{\Phi}(x, t) \hat{\omega}(x, t) \, dx \bigg|_{t=0}^{t=T} = \int_0^T \int_{\mathbb{R}^2} \frac{D\hat{\Phi}}{Dt} \hat{\omega} \, dx \, dt$$

for all $\hat{\Phi} \in C^1_c(\mathbb{R}^2 \times [0, T])$, which is exactly the definition of a weak solution.
Details of the change of variables Clearly \( \bar{\omega}_\varepsilon(x, t) = \tilde{\omega}_\varepsilon(y) \), and since \( \frac{\partial}{\partial t} R(-\alpha, t)x = \gamma \varepsilon \), we see that
\[
\partial_t \tilde{\phi}(x, t) = \partial_t \phi(y, t) - \alpha \varepsilon \gamma \varepsilon y \cdot \nabla_y \phi(y, t)
\]
Finally, we claim that
\[
\tilde{u}_\varepsilon(x, t) \cdot \nabla_y \tilde{\phi}(x, t) = u_\varepsilon(y) \cdot \nabla_y \phi(y, t), \quad \text{for } u_\varepsilon(y) = -\nabla_y^\perp (G * \tilde{\omega}_\varepsilon)(y)
\]
A convenient way to verify this is to define
\[
\tilde{\psi}_\varepsilon(\cdot, t) = G * \tilde{\omega}_\varepsilon(\cdot, t), \quad \tilde{\psi}_\varepsilon = G * \tilde{\omega}_\varepsilon(\cdot).
\]
Then it is straightforward to check that
\[
\tilde{\psi}_\varepsilon(x, t) = \tilde{\psi}_\varepsilon(R(-\alpha, t)x) = \tilde{\psi}_\varepsilon(y).
\]
Also, the definition of \( K \) implies that
\[
\tilde{u}_\varepsilon(x, t) = -\nabla_y^\perp \tilde{\psi}_\varepsilon(x, t) = -\nabla_y^\perp \tilde{\psi}_\varepsilon(R(-\alpha, t)x) = R(\alpha, t) \nabla_y^\perp \tilde{\psi}_\varepsilon(y)
\]
Essentially the same computation shows that
\[
\nabla_y \tilde{\phi}(x, t) = R(\alpha, t) \nabla_y \phi(y, t).
\]
Then (62) follows easily from these facts. Since the change of variables is area-preserving, the claim follows directly from (61), (62).

The next lemma will complete the proof of Proposition 1.

**Lemma 7.** Assume that \( \omega_\varepsilon \in L^1(S_0) \) satisfies the hypotheses of Proposition 1, and let \( \tilde{\omega}_\varepsilon \) denote its \( M \)-fold symmetric extension to \( \mathbb{R}^2 \) as in (51).

Then \( \tilde{\omega} \) satisfies (59).

**Proof.** We will prove that (59) holds for \( \tilde{\phi} \) of the form
\[
\tilde{\phi}(x) = \sum_{m=1}^M \phi(R(\theta_m)x), \quad \text{for some } \phi \in C^2_c(S_0).
\]
This is convenient, since the variational problem that yields \( \omega_\varepsilon \) is formulated on \( S_0 \) rather than \( \mathbb{R}^2 \). Deducing the general case of (59) from this special case is left as an exercise (with hints); see the end of this section.

We thus fix \( \phi \in C^2_c(S_0) \) and define \( \tilde{\phi} \) as above. We plan to use \( \phi \) to construct a bespoke 1-parameter family of functions \( \omega_\varepsilon^\sigma \), depending smoothly on \( \sigma \), such that
\[
\omega_\varepsilon^\sigma \in \Sigma_\varepsilon \quad \text{for all } \sigma \text{ in a neighborhood of } 0.
\]
The maximality of \( \omega_\varepsilon \) will then imply that
\[
\frac{d}{d\sigma} \tilde{\varepsilon}(\omega_\varepsilon^\sigma) = 0.
\]
We will construct \( \omega_\varepsilon^\sigma \) in such a way that this identity will yield (59).

**Step 1.** To start the construction, for \( (x, \sigma) \in \mathbb{R}^2 \times \mathbb{R} \), define \( Y(x, \sigma) \) as the solution of the initial-value problem
\[
\frac{dY}{d\sigma}(x, \sigma) = \nabla^\perp \phi(Y(x, \sigma)), \quad Y(x, 0) = x.
\]

---

1Our convention is that if \( f \) is a scalar function, then \( \nabla f \) is a column vector. When necessary, we will write \( Df \) for the corresponding row vector, i.e., \( Df = (\nabla f)^T \). Recall that the chain rule states that \( D(f \circ g) = (Df \circ g) Dg \). Taking transposes, it has the form \( \nabla (f \circ g) = (Dg)^T \nabla f \circ g \).

The computation of \( \tilde{u}_\varepsilon \) uses the chain rule in this form, together with the fact that for a rotation matrix, \( R(\theta)^T = R(-\theta) \).
Since $\nabla^2 \phi$ is clearly divergence-free, $x \mapsto Y(x, \sigma)$ is an area-preserving diffeomorphism for every $\sigma$. Next, define

$$Z(x, \sigma) = Y(x, \sigma) + q(\sigma) \quad \text{for } q : \mathbb{R} \to \mathbb{R}^2 \text{ chosen below to satisfy (63).}$$

We will arrange that $q(0) = 0$ and hence that $Z(x, 0) = x$.

We write $Z^{-1}(\cdot, \sigma)$ to denote the inverse of $Z(\cdot, \sigma)$ with respect to the $x$ variable, so that

$$Z^{-1}(Z(x, \sigma), \sigma) = x \text{ for every } (x, \sigma).$$

Now let

$$\omega^{\sigma}_{\varepsilon}(x) := \omega_{\varepsilon}(Z^{-1}(x, \sigma)).$$

We wish to arrange that $\omega^{\sigma}_{\varepsilon} \in \Sigma_{\varepsilon}$ for $\sigma$ in a neighborhood of 0. It is clear that $\omega^{\sigma}_{\varepsilon}$ satisfies

$$0 \leq \omega^{\sigma}_{\varepsilon} \leq \frac{1}{\pi \varepsilon^2}, \quad \int_{S_0} \omega^{\sigma}_{\varepsilon} \, dx = 1 \text{ for } \sigma \text{ sufficiently small.}$$

Moreover, since $\text{supp}(\omega_{\varepsilon})$ is a compact subset of $S_1$, the same holds for $\omega^{\sigma}_{\varepsilon}$ for all $\sigma$ in a neighborhood of 0. To complete the verification of (63), we must choose $q(t)$ so that $q(0) = 0$ and

$$\int_{S_0} |x|^2 \omega^{\sigma}_{\varepsilon}(x) \, dz = 1$$

for $\sigma$ close to 0. We use the (area-preserving) change of variables $y = Z^{-1}(x, \sigma)$ to compute

$$\int_{S_0} |x|^2 \omega^{\sigma}_{\varepsilon}(x) \, dx = = \int_{S_0} |x|^2 \omega_{\varepsilon}(Z^{-1}(x, \sigma)) \, dx$$

$$= \int_{S_0} |Z(y, \sigma)|^2 \omega_{\varepsilon}(y) \, dy$$

$$= \int_{S_0} |Y(y, \sigma) + q(\sigma)|^2 \omega_{\varepsilon}(y) \, dy$$

$$= \int_{S_0} |Y(y, \sigma)|^2 \omega_{\varepsilon}(y) \, dy + 2q(\sigma) \cdot \int_{S_0} Y(y, \sigma) \omega_{\varepsilon}(y) \, dy + |q(\sigma)|^2$$

Let

$$I_0(\sigma) := \int_{S_0} |Y(y, \sigma)|^2 \omega_{\varepsilon}(y) \, dy, \quad I_1(\sigma) := \int_{S_0} Y(y, \sigma) \omega_{\varepsilon}(y) \, dy$$

Since $\omega_{\varepsilon}$ is nonnegative and supported in $S_1$, it is clear that

$$I_1(0) = \int_{S_0} y \omega_{\varepsilon}(y) \, dy \neq 0.$$ 

We can thus arrange that $\int |x|^2 \omega^{\sigma}_{\varepsilon}(x) \, dz = 1$ for $|\sigma|$ small by setting

$$q(\sigma) := r(\sigma) I_1(\sigma), \quad \text{where } r(\sigma) \text{ solves } r^2 + 2r + \frac{I_0(\sigma) - 1}{|I_1(\sigma)|^2} = 0.$$

We choose the larger root, leading to

$$r(\sigma) = -1 + \sqrt{1 + \frac{1 - I_0(\sigma)}{|I_1(\sigma)|^2}} \approx \frac{1 - I_0(\sigma)}{2|I_1(\sigma)|^2} = O(\sigma) \text{ for } \sigma \approx 0,$$

since $I_0(0) = 1$. With this choice, (63) holds as required.
We also remark for future reference that
\[ I_0'(0) = 2 \int_{S_0} Y(y, 0) \cdot \frac{\partial Y}{\partial \sigma}(y, 0) \omega_\varepsilon(y) \, dy = 2 \int_{S_0} y \cdot \nabla^\perp \phi(0) \omega_\varepsilon(y) \, dy \]
and thus
\[ q'(0) = -\frac{I_0'(0)}{2|I_1(0)|^2} I_1(0) = -\frac{1}{|I_1(0)|^2} \left( \int_{S_0} y \cdot \nabla^\perp \phi(y) \, dy \right) I_1(0). \]

**Step 2.** Changing variables as above, we find that
\[
\tilde{\mathcal{E}}(\omega_\varepsilon^\sigma) = \frac{1}{2} \int_{S_0 \times S_0} G_M(x, x') \omega_\varepsilon^\sigma(x) \omega_\varepsilon^\sigma(x') \, dx \, dx' \\
= \frac{1}{2} \int_{S_0 \times S_0} G_M(Z(y, \sigma), Z(y', \sigma)) \omega_\varepsilon(y) \omega_\varepsilon(y') \, dy \, dy'.
\]
Thus (using symmetry at certain points to simplify things)
\[ 0 \overset{\text{6.1.1}}{=} \frac{d}{d\sigma} \tilde{\mathcal{E}}(\omega_\varepsilon^\sigma) \bigg|_{\sigma=0} \]
\[ = \int_{S_0 \times S_0} \nabla_y G_M(y, y') \cdot \frac{dZ}{d\sigma}(y, 0) \omega_\varepsilon(y) \omega_\varepsilon(y') \, dy \, dy' \\
= \int_{S_0 \times S_0} \nabla_y G_M(y, y') \cdot (\nabla^\perp \phi(y) + q'(0)) \omega_\varepsilon(y) \omega_\varepsilon(y') \, dy \, dy'.
\]
In general \( v \cdot w^+ = -v^+ \cdot w \), and it is clear that
\[
\int_{S_0} \nabla_y G_M(y, y') \omega_\varepsilon(y') \, dy' = \nabla_y \int_{S_0} G_M(y, y') \omega_\varepsilon(y') \, dy' = \nabla_y (G * \tilde{\omega}_\varepsilon(y))
\]
Using this and the computation of \( q'(0) \) from above, we deduce that
\[
0 = \int_{S_0} -\nabla^\perp (G * \omega) \cdot \nabla \phi \tilde{\omega}_\varepsilon \, dy \\
+ \left( \int_{S_0 \times S_0} \nabla_y G_M(y, y') \omega_\varepsilon(y) \omega_\varepsilon(y') \, dy \, dy' \right) \cdot \frac{I_1(0)}{|I_1(0)|^2} \left( \int_{S_0} y \cdot \nabla^\perp \phi(y) \, dy \right)
= \int_{S_0} -\nabla^\perp \left( (G * \omega_\varepsilon) + \frac{\alpha}{2} |y|^2 \right) \cdot \nabla \phi(y) \omega_\varepsilon(y) \, dy
\]
for
\[ \alpha = \frac{I_1(0)}{|I_1(0)|^2} \int_{S_0 \times S_0} \nabla_y G_M(y, y') \omega_\varepsilon(y) \omega_\varepsilon(y') \, dy \, dy'. \]
Writing \( \tilde{\psi}_\varepsilon = G * \tilde{\omega}_\varepsilon \) as above, we conclude by using symmetry considerations to deduce that
\[
0 = \int_{\mathbb{R}^2} -\nabla^\perp (\tilde{\psi}_\varepsilon) + \frac{\alpha}{2} |y|^2 \cdot \nabla \phi(y) \tilde{\omega}_\varepsilon(y) \, dy
\]
\[ \square \]
1.2.1. Exercises: fill in the missing part of the proof. In the proof of Proposition 1, we verified identity (59) only for $\hat{\phi}$ of the form

$$\hat{\phi}(x) = \sum_{m=1}^{M} \phi(R(\theta_m)x), \quad \text{for some } \phi \in C^2_c(S_0).$$

The following exercises guide you through a proof that this in fact implies that (59) holds for all $\phi \in C^2_c(R^2)$.

(Once this is known, a standard density argument shows that (59) holds for all $\tilde{\phi} \in C^1_c([R^2])$, which is the conclusion we needed.)

**Exercises.**

1. Verify by a change of variables that for $\hat{\phi} \in C^2_c$,

$$\int_{R^2} \tilde{\omega}_\varepsilon(y) \nabla \cdot (\tilde{\psi}_\varepsilon(y) + \frac{\alpha_\varepsilon}{2}|y|^2) \cdot \nabla \hat{\phi}(y) \, dy = \int_{R^2} \tilde{\omega}_\varepsilon(y) \nabla \cdot (\tilde{\psi}_\varepsilon(y) + \frac{\alpha_\varepsilon}{2}|y|^2) \cdot \nabla \hat{\phi}(R(\theta_m)y) \, dy$$

for every $m \in \{1, \ldots, M-1\}$.

2. Prove that for $\hat{\phi} \in C^2_c$,

$$\int_{R^2} \tilde{\omega}_\varepsilon(y) \nabla \cdot (\tilde{\psi}_\varepsilon(y) + \frac{\alpha_\varepsilon}{2}|y|^2) \cdot \nabla \hat{\phi}(y) \, dy = \int_{R^2} \tilde{\omega}_\varepsilon(y) \nabla \cdot (\tilde{\psi}_\varepsilon(y) + \frac{\alpha_\varepsilon}{2}|y|^2) \cdot \nabla \hat{\phi}_{\text{sym}}(y) \, dy$$

where

$$\hat{\phi}_{\text{sym}}(y) = \frac{1}{M} \sum_{m=0}^{M-1} \hat{\phi}(R(\theta_m)x).$$

3. Show that it suffices to prove the conclusion for $\hat{\phi} \in C^2_c$ with the symmetry

$$\hat{\phi}(R(\theta_m)x) = \hat{\phi}(x), \quad m \in \{1, \ldots, M-1\}.$$

4. Using properties of the support of $\tilde{\omega}_\varepsilon$, prove that, given any $\tilde{\phi}$ as in Exercise 3 above, there exists $\phi \in C^2_c(S_0)$ such that

$$\hat{\phi}(x) = \sum_{m=1}^{M} \phi(R(\theta_m)x) \quad \text{for all } x \in \text{supp}(\tilde{\omega}_\varepsilon)$$

5. Fill in any remaining details.

1.3. an important technical issue. The variational problem (53) that we consider is (mostly) invariant with respect to rotations: if $\omega_1 \in L^1(S_0)$ satisfies

$$0 \leq \omega_1 \leq \frac{1}{\pi \varepsilon^2} \quad \text{a.e.,} \quad \int_{R^2} \omega_1 \, dx = \int_{R^2} |x|^2 \omega_1 \, dx = 1,$$

then the same properties all hold for $\omega_2(x) := \omega_1(R(\theta)x)$ for any angle $\theta$. However, restrictions on the support of $\omega_1$ to sets such as $S_1$ are clearly not invariant under rotations. Since we want to find a maximizer that is compactly supported in $S_1$, this may cause problems. We therefore wish to fix these extra rotational degrees of freedom.

The following lemma will accomplish this. In it we write $\omega : S_1 \to \mathbb{R}$ as a function of polar coordinates $r, \theta$. Recall that in polar coordinates, $S_1 = \{(r, \theta) : \frac{1}{2} < r < 2, |\theta| < \frac{\pi}{2M}\}$. 


Lemma 8. Given a nonnegative \( \omega \in L^\infty(S_1) \), there exists a unique (up to sets of measure 0) function \( \omega^* \), called the \( \theta \)-symmetrization of \( \omega \), such that for almost every \( r \in (\frac{1}{2}, 2) \),

\[
\omega^*(r, \cdot) \text{ is a rearrangement } \omega(r, \cdot)
\]

and

\[
\omega^*(r, \theta) = \omega^*(r, -\theta), \quad \text{and } \theta \mapsto \omega^*(r, \theta) \text{ is nonincreasing for } \theta > 0.
\]

Moreover,

\[
E(\omega^*) \geq E(\omega).
\]

Note that it is clear that

\[
\int_{S_0} \omega^* \, dx = \int_{S_0} \omega \, dx, \quad \int_{S_0} |x|^2 \omega^* \, dx = \int_{S_0} |x|^2 \omega \, dx.
\]

It is also easy to see that if \( \text{supp}(\omega) \subset \bar{S}_1 \) then the same is true of \( \text{supp}(\omega^*) \). Thus \( \theta \)-symmetrization preserves all constraints in our variational problem. In considering the maximization problem \([53]\), we can therefore restrict our attention to \( \theta \)-symmetrized functions.

Proof. We only sketch the proof of the existence and uniqueness of \( \omega^* \). One approach is to prove the result first under the assumption that \( \omega \) is the characteristic function of a set, in which case a quite explicit formula for \( \omega^* \) can be given. The general case follows from the case of a characteristic function by defining \( \omega^* \) to be the function such that for every positive \( \lambda \), the set \( \{(r, \theta) : \omega^*(r, \theta) > \lambda\} \) is the \( \theta \)-symmetrization of \( \{(r, \theta) : \omega(r, \theta) > \lambda\} \), and noting that this uniquely determines \( \omega^* \).

To prove \([67]\), first note that in terms of polar coordinates, writing \( x \equiv (r, \theta) \) and \( x' \equiv (r', \theta') \),

\[
|x - x'| = (r^2 + r'^2 - 2rr' \cos(\theta - \theta'))^{1/2}.
\]

It follows from this and \([54]\), \([55]\) that

\[
\tilde{E}(\omega) = \int G_{M, \text{pol}}(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') r' \, dr' \, d\theta' \, dr \, d\theta
\]

for

\[
G_{M, \text{pol}}(r, r', \xi) = \sum_{m=0}^{M-1} \frac{1}{2\pi} \log \left[\left(r^2 + r'^2 - 2rr' \cos(\xi - \theta_m)\right)^{-1/2}\right]
\]

A basic theorem (the Riesz rearrangement inequality) implies that to prove \([67]\), it suffices to verify that

\[
G(r, r', -\xi) = G(r, r', \xi), \quad \text{and } \partial_\xi G_{M, \text{pol}}(r, r', \xi) \leq 0
\]

for \( 0 \leq \xi \leq \pi/M \). To do this we compute

\[
\partial_\xi G_{M, \text{pol}}(r, r', \xi) = -\frac{1}{4\pi} \sum_{m=0}^{M-1} \frac{2rr' \sin(\xi - \theta_m)}{r^2 + r'^2 - 2rr' \cos(\xi - \theta_m)}.
\]
If we fix $\eta > 0$ such that $\cosh \eta = (r^2 + r'^2)/2rr'$, then (writing out the definition of $\theta_m$)

$$\partial_{\xi} G_{M,pol}(r,r',\xi) = -\frac{1}{4\pi} \sum_{m=0}^{M-1} \frac{\sin(\xi - 2\pi m/M)}{\cosh \eta - \cos(\xi - 2\pi m/M)}.$$

Appealing to an explicit formula for the sum on the right (see formula (41.2.16) in *A Table of Series and Products* by E. Hansen, Prentice-Hall 1975) we find that

$$\partial_{\xi} G_{M,pol}(r,r',\xi) = -\frac{1}{4\pi} \frac{M \sin M\xi \cosh M\eta - \cos M\xi}{M \sin M\xi \cosh M\eta - \cos M\xi},$$

which is clearly negative for $0 \leq \xi \leq \pi/M$. □

1.4. Existence of maximizers.

**Proposition 2.** For every sufficiently small $\varepsilon > 0$, there exists $\omega_\varepsilon \in \Sigma_\varepsilon$ such that

$$\tilde{E}(\omega_\varepsilon) \geq \tilde{E}(\omega) \quad \text{for all } \omega \in \Sigma_\varepsilon.$$

Moreover, there exist $\alpha_\varepsilon, \mu_\varepsilon \in \mathbb{R}$ such that

$$(70) \quad \omega_\varepsilon = \frac{1}{\pi \varepsilon^2} 1_{A_\varepsilon}, \quad \text{for } A_\varepsilon := \{x \in S_1 : \psi_\varepsilon(x) + \mu_\varepsilon + \frac{\alpha_\varepsilon}{2}|x|^2 > 0\}$$

where

$$\psi_\varepsilon := G * \tilde{\omega}_\varepsilon.$$

Finally $\omega_\varepsilon$ is $\theta$-symmetrized, and $A_\varepsilon$ has the form

$$A_\varepsilon = \{(r,\theta) \in S_1 : |\theta| < \Theta(r)\}$$

for a function $\Theta$ that is $C^1$ wherever it is positive.

1.4.1. Strategy. Before sketching the proof, we discuss some of the issues:

The most common strategy for proving existence of solutions of variational problems is the *Direct Method in the Calculus of Variations*. In our context this is

1. Consider a sequence, say $(\omega_k)_{k=1}^\infty$ in $\Sigma_\varepsilon$ such that

$$\tilde{E}(\omega_k) \to \sup_{\Sigma_\varepsilon} \tilde{E}.$$

(For this we need $\Sigma_\varepsilon \neq \emptyset$, which implies that $\varepsilon$ cannot be too large.)

2. Establish uniform bounds on some norms of $\omega_k$. In our setting, both $L^\infty$ bounds and control of the support follow from the definition of $\Sigma_\varepsilon$

3. Use the bounds to deduce that the sequence $(\omega_k)_{k=1}^\infty$ has a subsequence that converges to some limit, possibly in a weak topology. In the present setting, it follows from the Banach-Alaoglu Theorem and the bounds established above that there is a subsequence, again denoted (after relabelling\(^2\)) $(\omega_k)_{k=1}^\infty$ that converges to a limit $\omega$ in the sense that

$$\int_{S_1} f\omega_k \, dx \to \int_{S_1} f\omega \, dx \quad \text{for all } f \in L^1(S_1),$$

that is, in the weak-* $L^\infty$ topology.

---

\(^2\)This is a common abuse of notation. To prevent a proliferation of subscripts, we imagine that we have passed to a subsequence $(\omega_k)_{k=1}^\infty$, then relabelled the subsequence so that it is now written as $(\omega_k)_{k=1}^\infty$. 
4. Verify that the limit \( \omega \) satisfies all relevant constraints and that it maximizes the energy. For the last point, it is enough to prove that

\[
\tilde{\mathcal{E}}(\omega) \geq \limsup \tilde{\mathcal{E}}(\omega_k).
\]

The above approach is direct and straightforward and can in fact be used to prove the existence of maximizers for the variational problem \( \tilde{\mathcal{E}} \). Unfortunately, having found a maximizer, it’s hard to prove things about it, such as conclusions \( (70) \). The problem is that we can only get information about a solution of a constrained variational problem by comparing it to “competitors”: other functions that satisfy the same constraints. For problem \( \tilde{\mathcal{E}} \), there are enough constraints that it is hard to find enough competitors, and therefore hard to extract useful information.

Since we need facts like \( (70) \) to finish the proof of Theorem 2, we are forced to use a more roundabout argument to prove Proposition 2.

The strategy followed by Turkington in his 1985 paper (following other work of various authors on related problems) is as follows:

1. First introduce an auxiliary functional

\[
\tilde{\mathcal{E}}_{\epsilon,p}(\omega) := \tilde{\mathcal{E}}(\omega) - \frac{\lambda_\epsilon}{p} \int_{S_1} \left( \frac{\omega(x)}{\lambda_\epsilon} \right)^p \, dx, \quad \lambda_\epsilon = \frac{1}{\pi \epsilon^2}
\]

and consider the auxiliary problem of maximizing \( \tilde{\mathcal{E}}_{\epsilon,p} \) in \( \Sigma_0 \cap L^p(S_1) \) in the set

\[
\Sigma_0 := \left\{ \omega \in L^p(S_1) : 0 \leq \omega \ \text{a.e.,} \quad \int_{S_1} \omega \, dx = \int_{S_1} \omega(x)|x|^2 \, dx = 1 \right\}.
\]

Using the Direct Method, one can prove the existence of a maximizer for every \( p > 2 \) denoted \( \omega_{\epsilon,p} \). Importantly, one can also derive conditions the maximizer satisfies, which is easier for \( \tilde{\mathcal{E}}_{\epsilon,p} \) than for \( \tilde{\mathcal{E}} \) with the \( L^\infty \) constraint. The equation we will find is

\[
\left( \psi_{\epsilon,p}(x) + \frac{\alpha_{\epsilon,p}}{2} |x|^2 + \mu_{\epsilon,p} \right)^+ = \left( \frac{\omega_{\epsilon,p}(x)}{\lambda_\epsilon} \right)^{p-1}
\]

2. Use the equations satisfied by \( \omega_{\epsilon,p} \) to derive uniform (with respect to \( p \)) bounds on \( \omega_{\epsilon,p} \).

3. Finally, use these uniform bounds to show that after passing to a subsequence,

\[
\omega_{\epsilon,p} \to \text{a limit } \omega_\epsilon \quad \text{as } p \to \infty
\]

in a suitable topology. One can then verify that \( \omega_\epsilon \) maximizes the original problem \( \tilde{\mathcal{E}} \) and derive conditions \( (70) \) from properties of \( \omega_{\epsilon,p} \).

We will not present the full details, but here are some key points.

### 1.4.2. Existence of maximizer \( \omega_{\epsilon,p} \) of \( \tilde{\mathcal{E}}_{\epsilon,p} \) for \( p \geq 3 \).

First, we claim that exist positive constants \( c_0, c_1 \) such that if \( p \geq 3 \) the for any \( \omega \in L^p(S_1) \)

\[
\sup_{x \in S_1} |G \ast \tilde{\omega}(x)| \leq c_0 \|\omega\|_{L^p}
\]

\[
\sup_{x \in S_1} |\nabla G \ast \tilde{\omega}(x)| \leq c_1 \|\omega\|_{L^p}
\]
These will be used several times. (Estimates of this sort hold for all p \(>2\), but they are not uniform: \(c_1 \not\rightarrow +\infty\) as \(p \searrow 2\).) Indeed, fix \(x \in S_1\) and note that since \(\tilde{\omega}\) is supported in \(B_2(0)\),

\[
G \ast \tilde{\omega}(x) = \int_{B_2(0)} \frac{1}{2\pi} \log\left(\frac{1}{|x-y|}\right) \tilde{\omega}(y) \, dy.
\]

Then (74) follows from Hölder’s inequality, since \(\text{supp}(\tilde{\omega}) \subset B_2(0)\),

\[
\|\log(x-\cdot)\|_{L^q(B_2(0))} \leq \|\log\|_{L^q(B_2(0))} < \infty \text{ for every } q < \infty.
\]

Similarly, \(\|\nabla G\|_{L^q(B_2(0))} < \infty\) for every \(q < 2\), leading by the same argument to (73).

It follows from (72) that

\[
\tilde{\mathcal{E}}(\omega) \leq C_0 + \frac{1}{p} \int_{S_1} \left(\frac{\omega}{\lambda_\omega}\right)^p \, dx
\]

for \(\omega \in \Sigma_0^\varepsilon\). (This is where the assumption \(p \geq 2\) is used.) It follows that

\[
\tilde{\mathcal{E}}_{\varepsilon,p}(\omega) + \frac{1}{p} \int_{S_1} \left(\frac{\omega}{\lambda_\omega}\right)^p \, dx \leq C_\varepsilon \quad \text{for } \omega \in \Sigma_0^\varepsilon.
\]

We may thus choose a sequence \((\omega_k)_{k=0}^\infty\) in \(\Sigma_0^\varepsilon\) such that

\[
\tilde{\mathcal{E}}_{\varepsilon,p}(\omega_k) \to \sup_{\Sigma_0^\varepsilon} \tilde{\mathcal{E}}_{\varepsilon,p},
\]

and it follows from (74) that \(\|\omega_k\|_{L^p}\) is uniformly bounded. One can then extract a subsequence (still labelled \((\omega_k)\)) that converges to a limit \(\omega_{\varepsilon,p}\) weakly in \(L^p\), i.e.

\[
\int_{S_1} f \omega_k \, dx \to \int_{S_1} f \omega_{\varepsilon,p} \, dx \quad \text{for all } f \in L^q(S_1), \quad \frac{1}{p} + \frac{1}{q} = 1.
\]

This convergence is strong enough to guarantee that \(\omega_{\varepsilon,p} \in \Sigma_0^\varepsilon\), and also that \(\tilde{\mathcal{E}}(\omega_k) \to \tilde{\mathcal{E}}(\omega_{\varepsilon,p})\). Standard properties of weak convergence imply that

\[
\|\omega_{\varepsilon,p}\|_{L^p} \leq \liminf_{k \to \infty} \|\omega_k\|_{L^p},
\]

and it follows that \(\tilde{\mathcal{E}}_{\varepsilon,p}(\omega_{\varepsilon,p}) \geq \lim_{k \to \infty} \tilde{\mathcal{E}}_{\varepsilon,p}\). Thus \(\omega_{\varepsilon,p}\) is the maximizer we seek. The argument also implies that

\[
\|\omega_{\varepsilon,p}\|_{L^p} \leq C_\varepsilon \quad \text{for all } p \geq 3.
\]

Finally, in view of Lemma 8, we may assume that \(\omega_{\varepsilon,p}\) is \(\theta\)-symmetrized.

1.4.3. Equations for the maximizer \(\omega_{\varepsilon,p}\).

We will argue as follows:

1. Find some way of constructing many \(\zeta \in L^\infty(S_1)\) such that

\[
\omega^\zeta := \omega_{\varepsilon,p} + \sigma \zeta \text{ satisfies all relevant constraints for } |\sigma| \ll 1.
\]

2. Then deduce equations from the condition (a consequence of the maximality of \(\omega_{\varepsilon,p}\))

\[
\frac{d}{d\sigma} \tilde{\mathcal{E}}_{\varepsilon,p}(\omega^\sigma) \bigg|_{\sigma=0} = 0 \quad \text{for all } \zeta \text{ as above.}
\]
3. If necessary we can extract additional information by constructing \( \zeta \in L^\infty(S_1) \) such that (76) holds only for all positive \( \sigma \ll 1 \). For such \( \zeta \) we only know that

\[
\frac{d}{d\sigma} \mathcal{E}(\omega_{\varepsilon,p} + \sigma \zeta) \leq 0,
\]

which still tells us something.

Here are the details:

1. The constraints we need to worry about in (76) are

\[
\omega^\sigma \geq 0, \quad \int_{S_1} \omega^\sigma \, dx = \int_{S_1} |x|^2 \omega^\sigma \, dx = 1.
\]

To satisfy these, \( \delta > 0 \) and

- consider \( \zeta_0 \in L^\infty(S_1) \) such that \( \text{supp}(\zeta) \subset \{ x \in S_1 : \omega_{\varepsilon,p}(x) \geq \delta \} \).
- Find \( \eta_1 \) and \( \eta_2 \) in \( L^\infty(S_1) \) with support in \( \{ x \in S_1 : \omega_{\varepsilon,p}(x) \geq \delta \} \), such that

\[
\int_{S_1} \eta_1 \, dx = 1, \quad \int_{S_1} |x|^2 \eta_1 \, dx = 0
\]

and

\[
\int_{S_1} \eta_2 \, dx = 0, \quad \int_{S_1} |x|^2 \eta_2 \, dx = 1.
\]

- Define

\[
\zeta = \zeta_0 + \eta_1 \left( \int_{S_1} \zeta_0 \, dx \right) + \eta_2 \left( \int_{S_1} |x|^2 \zeta_0 \, dx \right).
\]

Then (76) holds for \( |\sigma| \ll 1 \) (depending on \( \delta \) and the \( L^\infty \) norms of \( \zeta_0, \eta_1, \eta_2 \).)

2. Then (77) yields

\[
0 = \int_{S_1 \times S_1} G_M(x,x') \omega_{\varepsilon,p}(x') \zeta(x) \, dx \, dx' - \int_{S_1} \left( \frac{\omega_{\varepsilon,p}(x)}{\lambda_\varepsilon} \right)^{p-1} \zeta(x) \, dx.
\]

Writing \( \Psi_{\varepsilon,p} := \int_{S_1} G_M(x,x') \omega_{\varepsilon,p}(x') \zeta(x) \, dx' \) and expressing \( \zeta \) in terms of \( \zeta_0, \eta_1, \eta_2 \), this says that

\[
\int_{S_1} \left[ \Psi_{\varepsilon,p} - \left( \frac{\omega}{\lambda_\varepsilon} \right)^{p-1} \right] \zeta_0 \, dx + \mu_{\varepsilon,p} \int_{S_1} \zeta_0 \, dx + \alpha_{\varepsilon,p} \int_{S_1} \frac{|x|^2}{2} \zeta_0 \, dx = 0
\]

for

\[
\mu_{\varepsilon,p} = \int_{S_1} \left[ \Psi_{\varepsilon,p} - \left( \frac{\omega}{\lambda_\varepsilon} \right)^{p-1} \right] \eta_1 \, dx, \quad \alpha_{\varepsilon,p} = \int_{S_1} \left[ \Psi_{\varepsilon,p} - \left( \frac{\omega}{\lambda_\varepsilon} \right)^{p-1} \right] \eta_2 \, dx.
\]

Since \( \zeta_0 \) is an arbitrary function supported in \( \{ x : \omega_{\varepsilon,p}(x) > \delta \} \), and since \( \delta > 0 \) is arbitrary, we conclude that

\[
\Psi_{\varepsilon,p} + \frac{\alpha_{\varepsilon,p}}{2} |x|^2 + \mu_{\varepsilon,p} = \left( \frac{\omega_{\varepsilon,p}}{\lambda_\varepsilon} \right)^{p-1} \quad \text{in} \quad \{ x \in S_1 : \omega_{\varepsilon,p}(x) > 0 \}.
\]

3. Now we can repeat the argument with \( \zeta_0 \in L^\infty(S_1) \) being any nonnegative function supported in \( \{ x \in S_1 : \omega_{\varepsilon,p}(x) = 0 \} \), and \( \zeta \) defined as in (79), for some choice of \( \eta_1, \eta_2 \) as above. Then \( \omega^\sigma \) satisfies all the constraints when \( \sigma > 0 \), so we can apply (78) and argue as above to find that

\[
\Psi_{\varepsilon,p} + \frac{\alpha_{\varepsilon,p}}{2} |x|^2 + \mu_{\varepsilon,p} \leq 0 \quad \text{in} \quad \{ x \in S_1 : \omega_{\varepsilon,p}(x) = 0 \}.
\]
This completes the proof of (11).

### 1.4.4. Uniform bounds (independent of \( p \)) as \( p \to \infty \) on \( \omega_{\epsilon,p} \).

We claim that for all sufficiently small \( \epsilon \) there exists \( C_{\epsilon} \) such that

\[
\| \omega_{\epsilon,p} \|_{L^p} \leq C_{\epsilon} \quad \text{for all } p \geq 3.
\]

We know that

\[
\omega_{\epsilon,p} = \chi_{\epsilon}(\Phi_{\epsilon,p}^+) \frac{1}{p-1} \quad \text{for } \Phi_{\epsilon,p} := \psi_{\epsilon,p} + \frac{\alpha_{\epsilon,p}}{2}|x|^2 + \mu_{\epsilon,p}
\]

It also follows from (75) and (72), (73) that \( \|\psi_{\epsilon,p}\|_{W^{1,\infty}(S_1)} \leq C_{\epsilon} \). To prove (80), it therefore suffices to prove uniform (in \( p \), for fixed \( \epsilon \)) bounds for \( \alpha_{\epsilon,p} \) and \( \mu_{\epsilon,p} \).

The idea is to argue that if \( |\alpha_{\epsilon,p}| \) for example is too large, then since

\[
\alpha_{\epsilon,p} = \frac{1}{p} \partial_\nu(\Phi_{\epsilon,p} - \psi_{\epsilon,p}) \quad \text{and } \|\nabla \psi_{\epsilon,p}\|_{L^\infty} \leq C_{\epsilon},
\]

and using the assumption that \( \int |x|^2 \omega_{\epsilon,p} = \int \omega_{\epsilon,p} = 1 \), it must be the case that \( \Phi_{\epsilon,p} \) is large on a big subset of \( S_1 \). With (81), this could be used to show that \( \int \omega_{\epsilon,p} dx > 1 \), a contradiction. Bounds for \( |\mu_{\epsilon,p}| \) are proved in a similar way.

It follows from (80), elliptic regularity (the Calderon -Zygmund estimates) and Sobolev embedding theorems that for \( p \geq 3 \),

\[
\|\psi_{\epsilon,p}\|_{C^{1,1/3}(S_0)} \leq C \|\psi_{\epsilon,p}\|_{W^{2,3}(S_0)} \leq C \|\omega_{\epsilon,p}\|_{L^3(S_0)} \leq C \|\omega_{\epsilon,p}\|_{L^p(S_0)} \leq C_{\epsilon}.
\]

Here

\[
\|f\|_{C^{1,1/3}(S_0)} = \|f\|_{C^1(S_0)} + \sup_{x,y \in S_0} \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^{1/3}}.
\]

### 1.4.5. Limit as \( p \to \infty \) of \( \omega_{\epsilon,p} \) - conclusion of proof.

Using estimates (80) and (82), together with uniform bounds on \( \alpha_{\epsilon,p} \) and \( \mu_{\epsilon,p} \), we can send \( p \to \infty \) and find \( \omega_{\epsilon} \in L^\infty(S_1) \), \( \psi_{\epsilon} \in C^{1,1/3}(S_0) \) and numbers \( \alpha_{\epsilon}, \mu_{\epsilon} \) such that after passing to a subsequence,

\[
\omega_{\epsilon,p} \rightharpoonup \omega_{\epsilon} \quad \text{weak} \quad \text{in } L^\infty,
\]

\[
\alpha_{\epsilon,p} \to \alpha_{\epsilon}, \quad \mu_{\epsilon,p} \to \mu_{\epsilon}
\]

\( \psi_{\epsilon,p} \to \psi_{\epsilon} \) with respect to the \( C^1 \) norm

as \( p \to \infty \) (along the chosen subsequence). From this one can verify that \( \psi_{\epsilon} = G \ast \omega_{\epsilon} \). Using the fact that \( \omega_{\epsilon} \) is \( \theta \)-symmetrized one can check that

\[
\partial_\theta \psi_{\epsilon}(r, \theta) \begin{cases} 
< 0 & \text{if } 0 < \theta < \pi/2M \\
> 0 & \text{if } -\pi/2M < \theta < 0
\end{cases}
\]

We also have

\[
\omega_{\epsilon,p} = \lambda_{\epsilon} \left[ \left( \psi_{\epsilon,p} + \frac{\alpha_{\epsilon,p}}{2}|x|^2 + \mu_{\epsilon,p} \right)^+ \right]^{1/(p-1)}, \quad \lambda_{\epsilon} = \frac{1}{\pi \epsilon^2}.
\]

Using (83) and the \( C^1 \) convergence \( \psi_{\epsilon,p} \to \psi_{\epsilon} \), one can send \( p \to \infty \) in this identity to conclude that

\[
\omega_{\epsilon} = \frac{1}{\pi \epsilon^2} 1_{A_{\epsilon}} \quad \text{for } A_{\epsilon} := \{ x \in S_1 : \psi_{\epsilon}(x) - \mu_{\epsilon} - \frac{\alpha_{\epsilon}}{2}|x|^2 > 0 \}. 
\]
In particular, with this it is easy to verify that \( \omega_\varepsilon \) belongs to \( \Sigma_\varepsilon \), that is, it satisfies all the constraints. Then it is also straightforward to check that \( \omega_\varepsilon \) maximizes \( \tilde{\mathcal{E}} \) in \( \Sigma_\varepsilon \).

Finally, it follows from the (83), the Implicit Function Theorem, and the \( \theta \)-symmetry of \( \omega_\varepsilon \) and

\[
A_\varepsilon := \{x \in S_1 : \psi_\varepsilon(x) + \mu_\varepsilon + \frac{\alpha_\varepsilon}{2}|x|^2 > 0\}
\]

has the form

\[
\{ (r, \theta) \in S_1 : |\theta| < \Theta(r) \}
\]

for a function \( \Theta \) that is \( C^1 \) wherever it is positive.

This completes the (sketch of) the proof of Proposition 2.

1.5. compact support in \( S_1 \) of maximizers. The following result completes the proof of Theorem 2.

**Proposition 3.** Let \( \omega_\varepsilon \) be the maximizer found in Proposition 2. Then there exists \( \varepsilon_0 > 0 \) such that if \( 0 < \varepsilon < \varepsilon_0 \), then

\[
\text{supp}(\omega_\varepsilon) = \bar{A}_\varepsilon \subset S_1.
\]

Since \( S_1 \) is open, the proposition asserts that \( A_\varepsilon \) is a positive distance from \( \partial S_1 \).

We first prove some lemmas.

**Lemma 9.** Assume that \( 0 < \varepsilon < 1 \) and that \( \omega \in \mathbb{R}^\infty(\mathbb{R}^2) \) satisfies

\[
0 \leq \omega \leq \frac{1}{\pi \varepsilon^2}, \quad \int \omega = 1,
\]

\[
\int \log\left(\frac{1}{|x - x'|}\right) \omega(x) \omega(x') dx \geq \log \frac{1}{\varepsilon} - C_1.
\]

Then

\[
\inf_{x' \in \mathbb{R}^2 \setminus B_{\varepsilon}(x')} \omega(x) dx \leq \left( \frac{C_1 + \frac{1}{2}}{\log R} \right).
\]

**Proof.** It follows from (85), (86) that

\[
\int \frac{\varepsilon}{|x - x'|} \omega(x) \omega(x') dx \geq -C_1.
\]

Also, for every \( x \) and every \( R > 1 \)

\[
\int_{B(x, R \varepsilon)} \log\left(\frac{\varepsilon}{|x - x'|}\right) \omega(x') dx' \leq \int_{B(x, \varepsilon)} \log\left(\frac{\varepsilon}{|x - x'|}\right) \omega(x') dx' \leq \frac{1}{\pi \varepsilon^2} \int_{B(x, \varepsilon)} \log \frac{\varepsilon}{|x - x'|} dx' = \frac{1}{2}
\]

by an explicit computation. Thus

\[
\int_{|x - x'| \leq R \varepsilon} \log\left(\frac{\varepsilon}{|x - x'|}\right) \omega(x) \omega(x') dx \leq \frac{1}{2}.
\]

Subtracting this from (87), we find that

\[
-C_1 - \frac{1}{2} \leq \int_{|x - x'| > R \varepsilon} \log\left(\frac{\varepsilon}{|x - x'|}\right) \omega(x') \omega(x) dx
\]

\[
\leq \log \frac{1}{R} \int_{|x - x'| > R \varepsilon} \omega(x') \omega(x) dx'.
\]
Multiplying by \(-1\) and rewriting,
\[
\frac{(C_1 + \frac{1}{2})}{\log R} \geq \int_{|x-x'| \geq R} \omega(x) \omega(x') \, dx' \, dx = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} 1_{|x-x'| \geq R} \omega(x) \, dx \right) \omega(x') \, dx'.
\]

It follows that there exists some \(x' \in \mathbb{R}^3\) such that
\[
\int_{\mathbb{R}^2} 1_{|x-x'| \geq R} \omega(x) \, dx \leq \frac{(C_1 + \frac{1}{2})}{\log R}
\]
which is the conclusion of the lemma.

\[
\square
\]

**Lemma 10.** The maximizer \(\omega_\varepsilon\) found in Proposition 3 satisfies
\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} \log\left(\frac{1}{|x-x'|}\right) \omega_{\varepsilon}(x) \omega_{\varepsilon}(x') \, dx \, dx' \geq \log \frac{1}{\varepsilon} - C
\]
for some constant \(C\) depending only on \(M\).

In the proof we will follow the convention of writing \(C\) to mean a constant that could in principle be computed, and that may change from line to line. This may be confusing at first but it very convenient once one gets used to it.

**Proof.** For \(x, x' \in S_1\) and \(m \in \{1, \ldots, M-1\}\), there exists some constant \(C\), depending only on \(M\), such that \(\frac{1}{\varepsilon} \leq |x - R_m x'| \leq 4\). Thus, for any \(\mu \in L^1(S_1)\) and \(x \in S_1\),
\[
(88) \quad G * \tilde{\mu}(x) = \int_{S_1} \frac{1}{2\pi} \log\left(\frac{1}{|x-x'|}\right) \mu(x') \, dx' + \sum_{m=1}^M \int_{S_1} \frac{1}{2\pi} \log\left(\frac{1}{|x-R_m x'|}\right) \mu(x') \, dx'
\]
\[
(89) \quad = \int_{S_1} \frac{1}{2\pi} \log\left(\frac{1}{|x-x'|}\right) \mu(x') \, dx' + O(1).
\]

If \(\int_{S_1} \mu \, du = 1\), it follows that
\[
(90) \quad \check{\varepsilon}(\mu) = \int_{S_1} G * \tilde{\mu} \cdot \mu \, dx = \int_{S_1 \times S_1} \frac{1}{2\pi} \log\left(\frac{1}{|x-x'|}\right) \mu(x) \mu(x') \, dx \, dx' + O(1).
\]

It is easy to see that for \(\varepsilon\) small enough, there exists some \(p \in S_1\) such that
\[
\mu_\varepsilon := \frac{1}{\pi \varepsilon^2} 1_{B_\varepsilon(p)} \in \Sigma_\varepsilon
\]
(and in particular \(B_\varepsilon(p) \subseteq S_1\)). By an explicit computation
\[
(91) \quad \int_{S_1 \times S_1} \frac{1}{2\pi} \log\left(\frac{1}{|x-x'|}\right) \mu_\varepsilon(x) \mu_\varepsilon(x') \, dx \, dx' = \frac{1}{2\pi} \left( \log \frac{1}{\varepsilon} + \frac{1}{4} \right).
\]

Then since \(\omega_\varepsilon\) is a maximizer,
\[
\int_{S_1 \times S_1} \frac{1}{2\pi} \log\left(\frac{1}{|x-x'|}\right) \omega_\varepsilon(x) \omega_\varepsilon(x') \, dx \, dx' \geq \check{\varepsilon}(\omega_\varepsilon) - C \geq \check{\varepsilon}(\mu_\varepsilon) - C \geq \frac{1}{2\pi} \left( \log \frac{1}{\varepsilon} - C \right).
\]

\[
\square
\]
Lemma 11. Assume that \( \mu \) is a function such that
\[
0 \leq \mu \leq \frac{1}{\pi \epsilon^2}, \quad \int_{\mathbb{R}^2} \mu \, dx = \gamma.
\]
Then
\[
\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} G(x - y) \mu(y) \, dy \leq \frac{\gamma}{2\pi} \left( \log \frac{1}{\epsilon \sqrt{\gamma}} + \frac{1}{2} \right).
\]

Proof. Fix \( x \) and consider \( \mu_1 = \frac{1}{\pi \epsilon^2} \mathbf{1}_{B_{\sqrt{\gamma}}(x)} \), so that \( \int_{\mathbb{R}^2} \mu \, dx = \int_{\mathbb{R}^2} \mu_1 \, dx \).
Then it is straightforward to check that
\[
\int G(x - y) \mu_1(y) \, dy \leq \int G(x - y) \mu_1(y) \, dy,
\]
and an explicit computation shows that
\[
\int G(x - y) \mu_1(y) \, dy \leq \frac{\gamma}{2\pi} \left( \log \frac{1}{\epsilon \sqrt{\gamma}} + \frac{1}{2} \right).
\]

Proof of Proposition 3

Step 1. It follows from Lemma 9 with \( R = \epsilon^{-1/2} \) and Lemma 10 that there
exists some \( x_{\epsilon} \in S_1 \) such that
\[
\int_{B_{\sqrt{\gamma}}(x_{\epsilon})} \omega_{\epsilon}(x) \, dx \geq 1 - \frac{C}{\log \frac{1}{\epsilon}}, \quad \int_{S_1 \setminus B_{\sqrt{\gamma}}(x_{\epsilon})} \omega_{\epsilon}(x) \, dx \leq \frac{C}{\log \frac{1}{\epsilon}}.
\]
In addition, since \( \omega_{\epsilon} \) is \( \theta \)-symmetrized, we may assume that \( x_{\epsilon} \) lies on the positive
\( x \)-axis. For \( \epsilon \) small enough, it must be the case that
\[
B_{\sqrt{\gamma}}(x_{\epsilon}) \subset \{ x \in S_1 : \frac{7}{8} \leq |x|^2 \leq \frac{8}{7} \}
\]
since otherwise the condition \( \int_{S_1} |x|^2 \omega_{\epsilon} \, dx = 1 \) cannot hold.
If \( x \not\in B_{\sqrt{\gamma}}(x_{\epsilon}) \), then
\[
\psi_{\epsilon}(x) = G * \tilde{\omega}_{\epsilon}(x) \quad \text{and} \quad \psi_{\epsilon}'(x) = G * \tilde{\omega}_{\epsilon}'(x)
\]
where we have used Lemma 11 and \( \omega_{\epsilon} \) to estimate the integral over \( S_1 \setminus B_{\sqrt{\gamma}}(x_{\epsilon}) \)
in the last inequality.

Similar computations, using \( \omega_{\epsilon} \) and the fact that \( \log \frac{1}{|x - x'|} \geq \frac{1}{2} \log \frac{1}{\epsilon} - \log 2 \) for \( x, x' \in B_{\sqrt{\gamma}}(x_{\epsilon}) \), show that
\[
\psi_{\epsilon}(x) \geq \frac{1}{4\pi} \log \frac{1}{\epsilon} - C \quad \text{in } B_{\sqrt{\gamma}}(x_{\epsilon}).
\]
On the other hand, we know that

\[ \psi(x) = \frac{1}{\pi \varepsilon^2} 1_{\lambda} \text{ for } A_\varepsilon := \{ x \in S_1 : \Phi_\varepsilon > 0 \}, \]

where \( \Phi_\varepsilon := \psi(x) + \mu_\varepsilon + \frac{\alpha_\varepsilon}{2} |x|^2 > 0. \)

(Again we follow the convention of writing \( C \) to mean a constant that could in principle be computed, and \textit{that may change from line to line}.)

We will combine (94), (93) and (95) in various \textit{ad hoc} ways to extract increasing amounts of information about \( \alpha_\varepsilon \) and \( \text{supp}(\omega_\varepsilon) \), leading to the conclusion of the proposition. In doing so we will repeatedly use the obvious fact that for \( x \) and \( y \) in \( S_1 \),

\[ \text{Step 2.} \text{ We claim that it suffices to prove that there exists positive constants } \alpha_1 \text{ such that } \]

\[ |\alpha_\varepsilon| \leq C \text{ for all } \varepsilon \in (0, \varepsilon_1). \]

Indeed, assume that (97) holds and that \( 0 < \varepsilon < \varepsilon_1 \). Let \( x \) be any point in \( \text{supp}(\omega_\varepsilon) \), so that \( \Phi_\varepsilon(x) > 0 \), and let \( y \in B_{\sqrt{\varepsilon}}(x_\varepsilon) \) be a point where \( \Phi(x) \leq 0 \). It follows from (95) that \( L^2\{ y \in S_1 : \Phi_\varepsilon(y) > 0 \} = \pi \varepsilon^2 \), so such points certainly exist.

Then again applying (96) and using Claim 2, we find that

\[ \psi_\varepsilon(x) \geq \psi_\varepsilon(y) + \frac{\alpha_\varepsilon}{2} |y|^2 - |x|^2 \geq \psi_\varepsilon(y) - C \geq \frac{1}{4\pi} \log \frac{1}{\varepsilon} - C. \]

Then it follows from (93) that

\[ \frac{1}{2\pi} \log \left( \frac{1}{\text{dist}(x, B_{\sqrt{\varepsilon}})} \right) \geq \frac{1}{4\pi} \log \frac{1}{\varepsilon} - C. \]

From this we deduce that

if \( x \in \text{supp}(\omega_\varepsilon) \), then \( \text{dist}(x, B_{\sqrt{\varepsilon}}) \leq C \sqrt{\varepsilon} \),

which certainly implies the conclusion of the proposition.

\textbf{Step 3:} We now prove (97). For \( k > 0 \), define

\[ R_k = \{ (x_1, x_2) : |x_1 - 1| < kr_0, |x_2| < kr_0 \} \]

where \( r_0 > 0 \) is small enough (depending on \( M \)) that \( R_3 \subset S_1 \) and \( \frac{3}{4} \leq |x| \leq \frac{3}{4} \) for all \( x \in R_3 \).

We also assume that \( \varepsilon \) is small enough that \( B_{\sqrt{\varepsilon}}(x_\varepsilon) \subset R_1 \). Then it follows from (93) that

\[ \text{smalloutR2} \]

\[ |\psi_\varepsilon| \leq C \text{ in } S_1 \setminus R_2. \]

We now consider two cases:

\textbf{Case 1.} \( \text{supp}(\omega_\varepsilon) \cap (R_3 \setminus R_2) \neq \emptyset. \)

Let \( x \in \text{supp}(\omega_\varepsilon) \cap (R_3 \setminus R_2) \), so that \( \Phi_\varepsilon(x) > 0 \), and let \( y \) be any point in \( S_1 \setminus R_2 \) such that \( \Phi_\varepsilon(y) \leq 0 \). Then (98) implies that \( |\psi_\varepsilon(x)|, |\psi_\varepsilon(y)| \leq C \), and hence that

\[ |\psi_\varepsilon| \leq C \text{ in } S_1 \setminus R_2. \]

\textbf{Case 2.} \( \text{supp}(\omega_\varepsilon) \cap (R_3 \setminus R_2) = \emptyset. \)

Let \( x \in \text{supp}(\omega_\varepsilon) \cap (R_3 \setminus R_2) \), so that \( \Phi_\varepsilon(x) > 0 \), and let \( y \) be any point in \( S_1 \setminus R_2 \) such that \( \Phi_\varepsilon(y) \leq 0 \). Then (98) implies that \( |\psi_\varepsilon(x)|, |\psi_\varepsilon(y)| \leq C \), and hence that

\[ \alpha_\varepsilon |y|^2 - |x|^2 \leq C. \]
The choice of \( R_3 \) implies that \( \frac{3}{16} \leq |x|^2 \leq \frac{16}{3} \), and arguing as in Step 2, for \( \epsilon \) small enough there exist \( y, \tilde{y} \) such that

\[
\Phi_\epsilon(y_j) \leq 0 \quad \text{for} \quad j = 1, 2, \quad |y|^2 \leq \frac{1}{2}, \quad ||\tilde{y}||^2 \geq 2
\]

and thus \( |y|^2 - |x|^2 \leq -\frac{1}{3} \) and \( ||\tilde{y}||^2 - |x|^2 \geq \frac{2}{3} \). Employing these two choices in (99), we conclude that (97) holds.

**Case 2.** \( \text{supp} (\omega_\epsilon) \cap (R_3 \setminus R_2) = \emptyset \).

The definition (95) of \( \Phi_\epsilon \) implies that

\[
x_1 \alpha_\epsilon = \partial_1 (\Phi_\epsilon - \psi_\epsilon).
\]

We integrate this identity over \( R_2 \). Then

\[
\alpha_\epsilon \int_{R_2} x_1 \omega_\epsilon \, dx = \int_{R_2} \partial_1 \Phi_\epsilon \cdot \omega_\epsilon \, dx - \int_{R_2} \partial_1 \psi_\epsilon \cdot \omega_\epsilon \, dx.
\]

The assumption that \( \text{supp} (\omega_\epsilon) \cap (R_3 \setminus R_2) = \emptyset \) implies that \( \Phi_\epsilon \leq 0 \) in \( R_3 \setminus R_2 \). Thus \( \Phi_\epsilon^+ = 0 \) on \( \partial R_2 \). Integrating by parts in the \( x_1 \) variable, we thus see that

\[
\int_{R_2} \partial_1 \Phi_\epsilon \cdot \omega_\epsilon \, dx = \frac{1}{\pi \epsilon^2} \int_{R_2} (\partial_1 \Phi_\epsilon^+) \, dx = 0.
\]

Next we consider the second integral on the left-hand side of (100). It is useful to write

\[
\psi_\epsilon = \psi_{\epsilon,1} + \psi_{\epsilon,2}
\]

where

\[
\psi_{\epsilon,1}(x) = \int_{R_2} G(x - x') \omega_\epsilon(x') \, dx' \quad \text{and} \quad \psi_{\epsilon,2}(x) = \int_{R^2 \setminus R_3} G(x - x') \tilde{\omega}_\epsilon(x') \, dx'.
\]

Note that we have used the assumption that \( \text{supp} (\omega_\epsilon) \cap (R_3 \setminus R_2) = \emptyset \) to exclude \( R_3 \setminus R_2 \) from the domain of the integral that defines \( \psi_{\epsilon,2} \). For \( x \in R_2 \) and \( x' \in R^2 \setminus R_3 \), \( G(x - x') \) is bounded and smooth, so it follows that

\[
(101) \quad |\partial_1 \psi_{\epsilon,2}(x)| = \left| \int_{R^2 \setminus R_3} \partial_1 G(x - x') \tilde{\omega}_\epsilon(x') \, dx' \right| \leq C \quad \text{for} \quad x \in R_2.
\]

It follows that

\[
\left| \int_{R_2} \partial_1 \psi_{\epsilon,2} \omega_\epsilon \, dx \right| \leq C \quad \text{independent of} \quad \epsilon.
\]

In addition

\[
\partial_1 \psi_{\epsilon,2}(x) = \int_{R_2} \partial_1 \alpha_1 G(x - x') \omega_\epsilon(x') \, dx' = \frac{-1}{2\pi} \int_{R_2} \frac{x_1 - x'_1}{|x - x'|^2} \omega_\epsilon(x') \, dx'
\]

Thus

\[
\int_{R_2} \partial_1 \psi_{\epsilon,1} \omega_\epsilon \, dx = -\frac{1}{2\pi} \int_{R_2 \times R_2} \frac{x_1 - x'_1}{|x - x'|^2} \omega_\epsilon(x') \omega_\epsilon(x') \, dx \, dx'.
\]

Since the integrand is odd with respect to the symmetry \( (x, x') \mapsto (x', x) \), the integral clearly vanishes. Combining this fact with (101) and (100), we see that

\[
|\alpha_\epsilon| \leq \left( \int_{R_2} x_1 \omega_\epsilon \, dx \right)^{-1} C, \quad C \text{ independent of} \quad \epsilon.
\]

\( ^3 \)Strictly speaking we should check that it is permissible here to exchange differentiation and integration. For the particular kernel \( G(x) = -\frac{1}{2\pi} \log |x| \) this fact is rather standard.
Since $B_{\sqrt{\varepsilon}}(x_\varepsilon) \subset \mathbb{R}_2$, it follows from (92) that $\frac{1}{2} \leq \int_{\mathbb{R}_2} x_1 \omega_\varepsilon \, dx$ for $\varepsilon$ small. We thus conclude that $|\alpha_\varepsilon| \leq C$. In view of Step 2, this completes the proof of the Proposition. \qed

1.6. Some remarks.

1.6.1. general. The theorem presented above is just one example of using variational arguments to produce steady solutions of the Euler equations with vorticity behaving in a prescribed way.

Many results of this sort deal with vortex rings in 3d, a problem that is a little more technical (but still has a 2d character, as it generally reduces to problems for functions of two variables, that is, functions of cylindrical coordinates $(r, \theta, z)$ that are independent of $\theta$.) Other problems that can be approached by similar techniques involve traveling vortex dipoles.

Papers of this sort continue to be written today. There are also many open problems, some of which (roughly speaking, anything involving genuinely 3d behaviour) seem far out of reach at the moment.

For vortex rings, idea of maximizing energy with constraints on the rearrangement class is credited to T. Brooke Benjamin in the paper *The alliance of practical and analytical insights into the nonlinear problems of fluid mechanics*. Applications of methods of functional analysis to problems in mechanics (Joint Sympos., IUTAM/IMU, Marseille, 1975), pp. 8-29. Lecture Notes in Math., 503. Springer, Berlin, 1976. In this paper Benjamin refers to an earlier work of V. I. Arnold as the general inspiration for his method. (I have not yet been able to find a copy of Arnold’s paper.) He writes of Arnold’s approach:

*The general formalism of Arnold’s method is very difficult to apply, however, and from a practical point of view the perhaps most valuable aspect of Arnold’s discovery is simply that, for any specific problem, there will always be an underlying variational principle whose useful form may be worked out ad hoc.*

The “very difficult” judgment may not be universally shared, but the notion that “there will always be an underlying variational principle” is useful and interesting.

1.6.2. The approach of Benjamin and Burton. Benjamin’s approach, transposed to the problem discussed above, might be to obtain rigidly rotating solutions of Euler from the variational problem

\[ \text{maximize } \tilde{\mathcal{E}} \text{ in } \{ \omega \in L^1(S_0) : \omega \text{ is a rearrangement of } \eta, \int |x|^2 \omega(x) \, dx = 1 \} \]

where $\eta$ is some given nonnegative function, possibly smooth or continuous and certainly integrable.

We have seen in Proposition 1 that a maximizer should yield a rigidly rotating solution of the Euler equations. In fact, if we are able to prescribe $\eta$, then it yields a vast number of solutions.

One way to approach results in the spirit of Theorem 2 is to study this variational problem directly. Some of its features:

- we no longer confine the support of $\omega$ to the compact subset $\tilde{S}_1$. 

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4 modulo some technical details, since I’ve changed the variational problem a bit from that considered in Proposition 1. But there’s always a variational principle!
The constraint of minimizing within a rearrangement class is a strange one.

These make the problem hard – harder than appreciated by Benjamin in fact. The approach was rigorously implemented over a couple of decades by a group of researchers, notably G. Burton and collaborators. We will describe (without proof) some of these results.

First, changing notation from above, given an open \( \mathcal{D} \subset \mathbb{R}^N \) and a measurable function \( f : \mathcal{D} \rightarrow [0, \infty) \), we define \( f^* : [0, \infty) \rightarrow [0, \infty) \) to be the unique nonnegative decreasing function such that

\[
f^*(\lambda) := \mathcal{L}^N(\{x \in \mathcal{D} : f(x) \geq \lambda\})
\]

Then for any nonnegative measurable \( \eta \), we define

\[
\mathcal{R}(\eta) := \{\zeta : \mathcal{D} \rightarrow [0, \infty) \text{ measurable : } \zeta^* = \eta^*\}
\]

\[
\mathcal{RC}(\eta) := \{\zeta : \mathcal{D} \rightarrow [0, \infty) \text{ measurable : } \zeta^* = \eta^*1_{[0,\beta)} \text{ for some } \beta > 0\}
\]

\[
\mathcal{W}(\eta) := \{\zeta : \mathcal{D} \rightarrow [0, \infty) \text{ measurable : } \forall \lambda > 0, \int_{\mathcal{D}} (\zeta - \alpha)^+ \leq \int_{\mathcal{D}} (\eta - \alpha)^+\}.
\]

Note that \( \mathcal{R}(\eta) \) is just the set of rearrangements of \( \eta \).

\( \mathcal{RC}(\eta) \) is called the set of rearrangements of curtailments of \( \eta \).

Some relationships between these sets are summarized in the following.

**Theorem 3 (Douglas 1994).** Assume that \( \mathcal{L}^N(\mathcal{D}) = +\infty \) and that \( \eta_0 \in \mathcal{L}^p(\mathcal{D}) \) for some \( 1 < p < \infty \). Then

(i) \( \mathcal{W}(\eta_0) \) is a convex, weakly compact subset of \( \mathcal{L}^p(\mathcal{D}) \).

(ii) \( \mathcal{RC}(\eta_0) \) is the set of extreme points of \( \mathcal{W}(\eta_0) \).

(iii) \( \mathcal{W}(\eta_0) \) is the closed convex hull of \( \mathcal{R}(\eta_0) \).

(iv) \( \mathcal{R}(\eta_0) \) is weakly dense in \( \mathcal{W}(\eta_0) \).

We assume that \( \mathcal{L}^N(\mathcal{D}) = +\infty \) because the case of a set \( \mathcal{D} \) of finite measure is less relevant and interesting.

Versions of this theorem remain true if all the sets are intersected with sets of the form

\[
\{\eta : \mathcal{D} \rightarrow [0, \infty) : \int \eta \phi_i \, dx \leq a_i, i = 1, \ldots, J\}
\]

for some nonnegative functions \( \phi_i \) and constants \( a_i \).

It is also a general fact that a continuous, convex function on a closed compact set attains its maximum, and does so at an extreme point of the set.

Using these one can proceed as follows:

- verify that \( \mathcal{E} \) is continuous with respect to the \( \mathcal{L}^p \) weak-* topology on in the space

\[
\{\eta \in \mathcal{L}^p(S_0) : \int \eta \leq a, \int |x|^2 \eta \leq b\}
\]

- Then for any good enough \( \eta_0 \), \( \mathcal{E} \) attains its maximum in the set

\[
\{\omega \in \mathcal{W}(\eta_0) : \int |x|^2 \omega \leq a\}.
\]
• This maximum occurs at a point of
\[ \{ \omega \in \mathbb{R}C[\eta_0] : \int |x|^2 \omega \leq a \}. \]
• investigate conditions under which in fact the maximum occurs at a point in \( \mathbb{R}[\eta_0] \).

2. point vortex dynamics

Fix nonzero numbers \( \gamma_1, \ldots, \gamma_M \in \mathbb{R} \) and points \( p_1^0, \ldots, p_M^0 \in \mathbb{R}^2 \), and let \( p(t) = (p_1(t), \ldots, p_M(t)) : [0, T^*) \to (\mathbb{R}^2)^M \) solve

\[
\frac{dp_m}{dt} = \sum_{\ell \neq m} \frac{\gamma_\ell (p_m - p_\ell)}{2\pi |p_m - p_\ell|^2},
\]
(103)
for every \( m \in \{1, \ldots, M\} \). Assume also that \( p_m(t) \neq p_\ell(t) \) for all \( t \in [0, T^*) \) and \( m \neq \ell \).

We will not prove it, but for generic initial data one may take \( T^* = +\infty \). There are however initial data for which vortices collide in finite time, forcing \( T^* < \infty \).

**Theorem 4.** Let \( u_\varepsilon \in L^\infty(\mathbb{R}^2 \times [0, \infty), \mathbb{R}^2) \) be the unique (Yudovich) solution to the Euler equations with initial data \( u_\varepsilon^0 \) for which the vorticity \( \omega_\varepsilon^0 = \nabla \perp \cdot u_\varepsilon^0 \) belongs to \( L^\infty \) and satisfies

\[
\text{supp}(\omega_\varepsilon^0) \subset \bigcup_{m=1}^M B_\varepsilon(p_m^0), \quad \int_{B_\varepsilon(p_m^0)} \omega_\varepsilon^0 \, dx = \gamma_m, \quad \gamma_m \omega_\varepsilon^0(x) \geq 0 \text{ in } B_\varepsilon(p_m^0).
\]
(104)
Assume also that there exists some \( \alpha > 0 \) and \( p > 2 \) such that
\[ \| \omega_\varepsilon^0 \|_{L^p} \leq \varepsilon^{-\gamma}. \]
Then for any \( T < T^* \) and any \( \alpha < 1/2 \), there exist \( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon \leq \varepsilon_0 \),

\[
\text{supp}(\omega_\varepsilon(t)) \subset \bigcup_{m=1}^M B_\varepsilon(p_m(t)), \quad \int_{B_\varepsilon(p_m(t))} \omega_\varepsilon(x, t) \, dx = \gamma_m,
\]
(105)
for every \( t \in [0, T] \), where \( p_m(t) \) solves \( \text{(103)} \). Furthermore, \( \varepsilon_0 \) depends only on the initial data for \( \text{(103)} \), \( T, \gamma, \alpha \).

**Theorem 4.** Let \( u_\varepsilon \in L^\infty(\mathbb{R}^2 \times [0, \infty), \mathbb{R}^2) \) be the unique (Yudovich) solution to the Euler equations with initial data \( u_\varepsilon^0 \) for which the vorticity \( \omega_\varepsilon^0 = \nabla \perp \cdot u_\varepsilon^0 \) belongs to \( L^\infty \) and satisfies

\[
\text{supp}(\omega_\varepsilon^0) \subset \bigcup_{m=1}^M B_\varepsilon(p_m^0), \quad \int_{B_\varepsilon(p_m^0)} \omega_\varepsilon^0 \, dx = \gamma_m, \quad \gamma_m \omega_\varepsilon^0(x) \geq 0 \text{ in } B_\varepsilon(p_m^0).
\]
(104)
Assume also that there exists some \( \alpha > 0 \) and \( p > 2 \) such that
\[ \| \omega_\varepsilon^0 \|_{L^p} \leq \varepsilon^{-\gamma}. \]
Then for any \( T < T^* \) and any \( \alpha < 1/2 \), there exist \( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon \leq \varepsilon_0 \),

\[
\text{supp}(\omega_\varepsilon(t)) \subset \bigcup_{m=1}^M B_\varepsilon(p_m(t)), \quad \int_{B_\varepsilon(p_m(t))} \omega_\varepsilon(x, t) \, dx = \gamma_m,
\]
(105)
for every \( t \in [0, T] \), where \( p_m(t) \) solves \( \text{(103)} \). Furthermore, \( \varepsilon_0 \) depends only on the initial data for \( \text{(103)} \), \( T, \gamma, \alpha \).

The basic theorem is due to Marchioro and Pulvirenti (1983, 1994) with subsequent improvements by Caprini and Marchioro (2019) for example.

For initial data with vorticity in \( L^p(\mathbb{R}^2) \) for \( 1 \leq p < \infty \), solutions of the Euler equations are known to exist, if not necessarily to be unique. These not only satisfy a weak form of the Euler equations, they also have additional good properties. For example, the vorticity is transported by the flow, in a suitable sense. As a consequence of the fact that \( \varepsilon_0 \) depends only the size if the initial data only through its \( L^p \) norm, one could in fact prove that some version of the theorem remains true for solutions with the good properties mentioned above, if we assume only that the initial data belongs to \( L^p \) for some \( p > 2 \).

We will proceed as follows.

Let \( u_\varepsilon \) be the solution from the statement of the theorem.
Let $X_ε(α,t)$ be the associated particle trajectory map, solving
\[ \frac{dX_ε}{dt}(α, t) = u_ε(X_ε(α, t), t). \]
Define $A_ε(·, t) := X_ε(·, t)^{-1}$. Then the vorticity transport formula can be written
\[ \omega_ε(x, t) = \omega_ε^0(A_ε(x, t)) \]
For $m = 1, \ldots, M$, define
\[ \omega_ε^0(·, t) := \omega_ε^0 I_{B_r(p_ε)} \]
\[ \omega_ε(·, t) = \omega_ε^0(·, t)(A_ε(x, t)) \]
\[ u_ε(·, t) = K\omega_ε(·, t). \]
Then
\[ \omega_ε = \sum_{m=1}^{M} \omega_ε^0(·, t), \quad \text{and} \]
\[ u_ε = \sum_{m=1}^{M} u_ε^0(·, t), \]
and the Euler equations can be written as a system
\[ \partial_t \omega_ε + (u_ε + F_ε, m) \cdot \nabla \omega_ε = 0, \quad u_ε = K\omega_ε, \quad F_ε, m = \sum_{\ell \neq m} u_ε, \ell. \]

We can analyze this by temporarily forgetting the coupling and simply considering $F_ε, m$ to be some given function. This leads to the equation (dropping subscripts)
\[ \partial_t \omega + (u + F) \cdot \nabla \omega = 0, \quad u = K\omega, \quad F \text{ given.} \]
For the data we will consider, this describes a single vortex in a background flow described by the given vector field $F$.

**2.1. a single vortex in a background flow – vortex dynamics.** Assume $F : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^2$ is smooth and that
\[ \nabla \cdot F(·, t) = 0 \text{ for all } t, \]
\[ |F(x, t) - F(y, t)| \leq L|x - y| \text{ for all } x, y, t. \]
and consider the equation
\[ \partial_t \omega + (u + F) \cdot \nabla \omega = 0, \]
\[ u(·, t) = K\omega(·, t). \]

**Theorem 5.** Let $\omega_ε : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ solve (110) for initial data $\omega_ε^0 \in L^\infty(\mathbb{R}^2)$ that does not change sign. Let $Ω := \int_{\mathbb{R}^2} \omega_ε^0$ and define
\[ p_ε(t) = \frac{1}{Ω} \int_{\mathbb{R}^2} x\omega_ε(x, t) \, dx \]
\[ I_ε(t) = \frac{1}{2Ω} \int_{\mathbb{R}^2} |x - p_ε(t)|^2 \omega_ε(x, t) \, dx \]
Let $p(t)$ solve
\[ \frac{d}{dt} p(t) = F(p(t), t), \quad p(0) = p_ε(0). \]
Then
\[ I_ε(t) \leq e^{2lt} I_ε(0) \]
and

\begin{equation}
|p(t) - p_\varepsilon(t)| \leq e^{Lt} \left( |p_\varepsilon(0) - p(0)| + Lt \sqrt{2I(0)} \right).
\end{equation}

A couple of remarks:

First, it is easy to check that if \( \omega \) is a probability measure on a space \( X \), then the function

\[ I_\omega(y) := \frac{1}{2} \int_{x \in X} |x - y|^2 \, d\omega \]

is minimized by \( y = \int_{x \in X} x \, d\omega \). In particular,

\[ I_\varepsilon(t) \leq \frac{1}{2\Omega} \int_{\mathbb{R}^2} |x - y|^2 \, \omega_\varepsilon(x, t) \, dx \quad \text{for every } y \in \mathbb{R}^2. \]

Second, one can check that the vorticity transport formula for (110) is

\[ \omega(X(\alpha, t), t) = \omega(\alpha, 0), \quad \text{where } \frac{dX}{dt} = (u + F)(X(\alpha, t), t), \quad X(\alpha, 0) = \alpha. \]

**Proof.** First note that the antisymmetry of the Biot-Savart kernel implies that

\begin{equation}
\int_{\mathbb{R}^2} u_\varepsilon(x, t) \omega_\varepsilon(x, t) \, dx = \int_{\mathbb{R}^2 \times \mathbb{R}^2} K(x - y) \omega_\varepsilon(x) \omega_\varepsilon(y, t) \, dx \, dy = 0.
\end{equation}

Similarly, using antisymmetry and the fact that \( z \cdot K(z) = 0 \), we have

\begin{equation}
\int_{\mathbb{R}^2} x \cdot u(x, t) \omega_\varepsilon(x, t) \, dx = \int_{\mathbb{R}^2 \times \mathbb{R}^2} x \cdot K(x - y) \, \omega_\varepsilon(y, t) \, \omega_\varepsilon(x, t) \, dx
\end{equation}

\begin{equation}
= \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (x - y) \cdot K(x - y) \, \omega_\varepsilon(y, t) \, \omega_\varepsilon(x, t) \, dx
\end{equation}

\[ = 0. \]

We will use these below.

Next, for an arbitrary \( C^1 \) function \( \phi \),

\[ \frac{d}{dt} \int_{\mathbb{R}^2} \phi(x, t) \omega_\varepsilon(x, t) \, dx = \int_{\mathbb{R}^2} (\partial_1 \phi \, \omega_\varepsilon + \phi \partial_1 \omega_\varepsilon) \, dx \]

\begin{equation}
= \int_{\mathbb{R}^2} (\partial_1 \phi \, \omega_\varepsilon - \phi(u + F) \cdot \nabla \omega_\varepsilon) \, dx
\end{equation}

\[ = 0 \quad \text{by P.100.} \]

We use this identity to compute

\[ \frac{d}{dt} I_\varepsilon(t) = \frac{1}{\Omega} \int_{\mathbb{R}^2} (x - p_\varepsilon(t)) \cdot (-\frac{dp_\varepsilon}{dt} + u(x, t) + F(x, t)) \, \omega_\varepsilon(x, t) \, dx \]

\begin{equation}
= \frac{1}{\Omega} \int_{\mathbb{R}^2} (x - p_\varepsilon(t)) \cdot (F(x, t) - \frac{dp_\varepsilon}{dt}) \, \omega_\varepsilon(x, t) \, dy \, dx.
\end{equation}

The definition of \( p_\varepsilon(t) \) implies that for any vector \( v(t) \), independent of \( x \),

\[ \frac{1}{\Omega} \int_{\mathbb{R}^2} (x - p_\varepsilon(t)) \cdot v \, \omega_\varepsilon(x, t) \, dx = v \cdot \int_{\mathbb{R}^2} (x - p_\varepsilon(t)) \frac{\omega_\varepsilon(x, t)}{\Omega} \, dx = 0, \]
so it follows that
\[
\frac{d}{dt} I_\varepsilon(t) = \frac{1}{\Omega} \int_{\mathbb{R}^2} (x - p_\varepsilon(t)) \cdot (F(x, t) - F(p_\varepsilon(t), t)) \omega_\varepsilon(x, t) \, dx
\]
\[
\leq L \frac{1}{\Omega} \int_{\mathbb{R}^1} |x - p_\varepsilon(t)|^2 \omega_\varepsilon(x, t) \, dx = 2L I_\varepsilon(t).
\]

Then (113) follows from Grönwall’s inequality.

Next,
\[
\frac{d}{dt} p_\varepsilon(t) = \frac{1}{\Omega} \int_{\mathbb{R}^2} (u_\varepsilon(x, t) + F(x, t)) \omega_\varepsilon(x, t) \, dx
\]
\[
(117)
\]
\[
\frac{d}{dt} \omega_\varepsilon(x, t) = \frac{1}{\Omega} \int_{\mathbb{R}^2} F(x, t) \omega_\varepsilon(x, t) \, dx.
\]

Thus
\[
\frac{d}{dt} (p_\varepsilon(t) - p(t)) = \frac{1}{\Omega} \int_{\mathbb{R}^2} F(x, t) \omega_\varepsilon(x, t) \, dx - F(p(t), t)
\]
\[
= F(p_\varepsilon(t), t) - F(p(t), t) + \frac{1}{\Omega} \int_{\mathbb{R}^2} (F(x, t) - F(p_\varepsilon(t), t)) \omega_\varepsilon(x, t) \, dx.
\]

It follows that
\[
\frac{d}{dt} |p_\varepsilon(t) - p(t)| \leq L |p_\varepsilon(t) - p(t)| + \frac{1}{\Omega} \int_{\mathbb{R}^2} |x - p_\varepsilon(t)| \omega_\varepsilon(x, t) \, dx
\]
\[
\leq L |p_\varepsilon(t) - p(t)| + L \sqrt{2 I_\varepsilon(t)}
\]
using Jensen’s inequality at the last step. Then (113) implies that
\[
\frac{d}{dt} |p_\varepsilon(t) - p(t)| \leq L |p_\varepsilon(t) - p(t)| + L e^{Lt} \sqrt{2 I_\varepsilon(0)}.
\]

This implies (114), again by Grönwall’s inequality. Explicitly, by rearranging we obtain
\[
\frac{d}{dt} |p_\varepsilon(t) - p(t)| e^{-Lt} \leq L \sqrt{2 I_\varepsilon(0)}.
\]

We now obtain (114) by integrating. \(\square\)

Informally, the theorem suggests:

For initial data consisting of \(M\) well-separated blobs, as long as they remain well-separated, they remain concentrated, and they are governed by the right ODE.

We need them to remain well-separated so that the background flow generated by other vortices remains Lipschitz continuous, with control over the Lipschitz constant.

So in a way, the hard issue is one of dynamic stability of the set of functions of well-separated blobs.
2.2. a single vortex in a background flow – bounds on growth of support.

**Theorem 6.** Assume that $F$ satisfies [109], and let $\omega_\varepsilon : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ solve [110] for initial data $\omega^0_\varepsilon \in L^\infty(\mathbb{R}^2)$ that does not change sign, and such that
\[
supp(\omega^0_\varepsilon) \subset B_\varepsilon(p_\varepsilon(0)).
\]
Further assume that there exists $p > 2$ and $\gamma > 0$ such that
\[
\|\omega^0_\varepsilon\|_p \leq \varepsilon^{-\gamma}.
\]
Let $\Omega := \int_{\mathbb{R}^2} \omega$. Then for any $a < 1/2$ and $T > 0$, there exists $\varepsilon_0 > 0$, depending on $a, T, p, \gamma, \Omega, L$ such that if $0 < \varepsilon < \varepsilon_0$ then
\[
supp(\omega_\varepsilon(t)) \subset B_{\varepsilon^a}(p_\varepsilon(t)),
\]
where $p_\varepsilon(t)$ is defined by (111).

**Remark 3.** A natural choice of $\omega^0_\varepsilon$ is a function of the form
\[
\omega^0_\varepsilon(x) = \frac{1}{\varepsilon^2} \omega^0(\frac{x - x_0}{\varepsilon})
\]
for some fixed $\omega^0 \in L^p(\mathbb{R}^2)$ with support in the unit ball. This would lead to $\gamma = 2(1 - \frac{1}{p})$. But we allow $\gamma$ to be any positive number, a very mild hypothesis.

We will prove some lemmas in greater generality than is needed for the present result, as they may be used again in slightly different contexts.

The first lemma is responsible for the restriction to $L^p$ norms for $p > 2$ in the hypothesis of Theorem [6] and thus also in Theorem [7].

**Lemma 12.** Assume that $\omega \in L^p \cap L^1(\mathbb{R}^2)$ for some $p > 2$. Then for any $x \in \mathbb{R}^2$,
\[
\left| \int_{\mathbb{R}^2} K(x - y) \omega(y) \, dy \right| \leq \int_{\mathbb{R}^2} \frac{\|\omega(y)\|_p}{2\pi|x - y|} \, dy \leq C\|\omega\|_p^{\frac{p}{p-1}} \|\omega\|_1^{\frac{p-2}{p-1}}.
\]

**Proof.** The first inequality is obvious. To prove the second, consider any $R > 0$ and let $q = \frac{p}{p-1} < 2$, so that $\frac{1}{p} + \frac{1}{q} = 1$. Then
\[
\int_{\mathbb{R}^2} \frac{\|\omega(y)\|_p}{2\pi|x - y|} \, dy = \int_{B_R(x)} \frac{\|\omega(y)\|_p}{2\pi|x - y|} \, dy + \int_{B_R(x)^c} \frac{\|\omega(y)\|_p}{2\pi|x - y|} \, dy
\]
\[
\leq C \left( \int_{B_R(x)} |x - y|^{-q} \, dy \right)^{1/q} \|\omega\|_p + \frac{C}{R} \|\omega\|_1
\]
\[
= CR^{\frac{q}{q-1}}\|\omega\|_p + CR^{-1}\|\omega\|_1.
\]
The lemma follows by choosing $R = (\|\omega\|_1/\|\omega\|_p)^{q/2}$.

The next lemma is in some ways the crucial point in the proof of Theorem [6].

Given a vorticity distribution with center of vorticity $p_\varepsilon$ and a point $x$ a certain distance $R$ from $p_\varepsilon$, it estimates the radial (with respect to the center of vorticity) component at $x$ of the velocity generated by the part of the vorticity supported in a small ball $B_r(p_\varepsilon)$ with $r < R$. 

One can check that the rotational component is

\[
\begin{align*}
\text{rot.comp} & \quad (118) \quad \left| \frac{x-p_\varepsilon}{|x-p_\varepsilon|} \cdot \int_{B_r} \frac{(x-y)^\perp \omega_\varepsilon(y)}{|y-x|^2} \, dy \right| \lesssim \frac{1}{R} \frac{1}{\Omega} \int_{B_r} \omega(y) \, dy.
\end{align*}
\]

For example, a typical situation below will be \( R = r \equiv \varepsilon^a \) for \( a < 1/3 \), and \( I(\omega_\varepsilon) \equiv \varepsilon^2 \). In this case, the lemma and (118) (plus a few additional computations) show that

- rotational component of velocity \( \equiv \varepsilon^{-a} \gg 1 \),
- radial component of velocity \( \equiv \varepsilon^{2-3a} \ll 1 \).

So we obtain a really striking "improvement of smallness" by focussing on the radial component, which is all we will really need for the proof of the Theorem.

**Lemma 13.** Let \( \omega_\varepsilon : \mathbb{R}^2 \to [0, \infty) \) be a bounded, nonnegative function. Let \( \Omega = \int_{\mathbb{R}^2} \omega_\varepsilon(x) \, dx \) and define

\[
I(\omega_\varepsilon) := \frac{1}{\Omega} \int |x-p_\varepsilon|^2 \omega_\varepsilon(x) \, dx < \infty, \quad p_\varepsilon = \frac{1}{\Omega} \int_{\mathbb{R}^2} x \omega_\varepsilon(x) \, dx.
\]

Then for any \( x \neq p_\varepsilon \), if \( 0 < r < R := |x| \), then

\[
\left| \frac{|x-p_\varepsilon|}{|x|} \cdot \int_{B_r} \frac{(x-y)^\perp \omega_\varepsilon(y)}{|y-x|^2} \, dy \right| \leq \frac{C}{(R-r)^2} I(\omega_\varepsilon).
\]

**Proof.** First, we may assume that \( \Omega = 1 \) and \( p_\varepsilon = 0 \), since

- if \( \Omega \neq 1 \) we can replace \( \omega_\varepsilon \) by \( \tilde{\omega}_\varepsilon := \omega_\varepsilon/\Omega \) (and then drop the tilde).
- if \( p_\varepsilon \neq 0 \), we can change variables, setting \( \tilde{x} = x - p_\varepsilon \), \( \tilde{y} = y - p_\varepsilon \), and \( \tilde{\omega}_\varepsilon(\tilde{y}) = \omega_\varepsilon(y) \) (then again drop the tildes).

We thus assume that

\[
\int_{\mathbb{R}^2} \omega_\varepsilon(x) \, dx = 1, \quad \text{and} \quad p_\varepsilon = \int_{\mathbb{R}^2} x \omega_\varepsilon(x) \, dx = 0.
\]

Since \( x \cdot x^{\perp} = 0 \) and \( p_\varepsilon = 0 \),

\[
\begin{align*}
\int_{B_r} \frac{x}{|x|} \cdot \frac{(y-x)^{\perp}}{|y-x|^2} \omega_\varepsilon(y) \, dy & = \int_{B_r} \frac{x \cdot y^{\perp}}{|x|} \frac{1}{|y-x|^2} \omega_\varepsilon(y) \, dy - \frac{x}{|x|^2} \int_{\mathbb{R}^2} y^{\perp} \omega_\varepsilon(y) \, dy \\
& = \int_{B_r} \frac{x \cdot y^{\perp}}{|x|} \left( \frac{1}{|y-x|^2} - \frac{1}{|x|^2} \right) \omega_\varepsilon(y) \, dy - \int_{B_r} \frac{x \cdot y^{\perp}}{|x|^3} \omega_\varepsilon(y) \, dy \\
& = T_1 + T_2.
\end{align*}
\]

For \( y \in B_r \), clearly \( |2y-x| \leq 2|y| + |x| \leq 3R \) and \( |y-x| \geq R-r \), so

\[
\left| \frac{1}{|y-x|^2} - \frac{1}{|x|^2} \right| = \left| \frac{y \cdot (2y-x)}{|y-x|^2 |x|^2} \right| \lesssim \frac{3|y|}{(R-r)^2 R}.
\]

It follows that

\[
|T_1| \lesssim \frac{3}{h^2 R} \int_{\mathbb{R}^2} |y|^2 \omega_\varepsilon(y) \, dy = \frac{6}{(R-r)^2 R} I(\omega_\varepsilon)
\]
For $y \in B_r$, we have $|y| / r \geq 1$. Thus
\[
\left| \frac{x \cdot y}{|x|^3} \right| \leq \frac{|y|}{R^2} \leq \frac{|y|^2}{R^2r}.
\]
We deduce that
\[
|T_2| \leq \frac{1}{R^2r} I(\omega_\varepsilon),
\]
We obtain the conclusion of the lemma by adding the estimates of $T_1$ and $T_2$ and noting that
\[
\frac{1}{(R-r)^2R} + \frac{1}{R^2r} \leq \frac{1}{(R-r)^2R} + \frac{1}{R(R-r)r} = \frac{1}{(R-r)^2r}.
\]

**Lemma 14.** Assume that $\omega_\varepsilon$ satisfies the assumptions of Lemma 13 and that $F$ satisfies (109). Then for any $x \in \mathbb{R}^2$,
\[
\left| F(x) - \int_{\mathbb{R}^2} F(y) \frac{\omega_\varepsilon(y)}{\Omega} dy \right| \leq CL(|x - p_\varepsilon| + \sqrt{I(\omega_\varepsilon)}).
\]

**Proof.** We compute
\[
\left| F(x) - \int_{B_r} -F(y) \frac{\omega_\varepsilon(y)}{\Omega} dy \right| \leq |F(x) - F(p_\varepsilon)| + \left| \int_{B_r} F(p_\varepsilon) - F(y) \frac{\omega_\varepsilon(y)}{\Omega} dy \right|
\]
\[
\leq L|x - p_\varepsilon| + L \int |x - p_\varepsilon| \frac{\omega_\varepsilon(y)}{\Omega} dy
\]
\[
\leq L|x - p_\varepsilon| + \sqrt{2L} \left( \frac{|x - p_\varepsilon|^2 \omega_\varepsilon(y)}{2} \right)^{1/2}.
\]

**Lemma 15.** Assume that $G \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ satisfies
\[
G(x) = 0 \quad \text{in } B_r, \quad G(x) \cdot x^\perp = 0 \quad \text{everywhere}.
\]
Then
\[
\left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} G(x) \cdot \frac{(x - y)^\perp}{|x - y|^2} \omega_\varepsilon(x) \omega_\varepsilon(y) dx dy \right| \leq C \left( \frac{\|G\|_\infty}{R^3} + \frac{\|\nabla G\|_\infty}{R^2} \right) m(R) I(\omega_\varepsilon)
\]
where
\[
m(R) = \int_{B_r} \omega_\varepsilon(y) dy.
\]

**Proof.** Step 1. Using the antisymmetry of $(x - y)^\perp /|x - y|^2$,
\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} G(x) \cdot \frac{(x - y)^\perp}{|x - y|^2} \omega_\varepsilon(x) \omega_\varepsilon(y) dx dy
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (G(x) - G(y)) \cdot \frac{(x - y)^\perp}{|x - y|^2} \omega_\varepsilon(x) \omega_\varepsilon(y) dx dy.
\]
Let us write
\[
f(x, y) = \frac{(G(x) - G(y)) \cdot (x - y)^\perp}{2|x - y|^2} \omega_\varepsilon(y) \omega_\varepsilon(x).
\]
Our assumption about the support of $G$ implies that $f = 0$ in $B_R \times B_R$. We thus have

$$
\int_{\mathbb{R}^2 \times \mathbb{R}^2} G(x) \cdot \frac{(x-y)^\perp}{|x-y|^2} \omega_\varepsilon(x) \omega_\varepsilon(y) \, dx \, dy
= \int_{B_{\varepsilon} \times \mathbb{R}^2} f(x,y) \, dx \, dy + \int_{\mathbb{R}^2 \times B_{\varepsilon}} f(x,y) \, dx \, dy - \int_{B_{\varepsilon} \times B_{\varepsilon}} f(x,y) \, dx \, dy
= 2 \int_{B_{\varepsilon} \times \mathbb{R}^2} f(x,y) \, dx \, dy - \int_{B_{\varepsilon} \times B_{\varepsilon}} f(x,y) \, dx \, dy
$$

by symmetry. Thus the quantity we seek to estimate may be written as

$$
2 \int_{B_{\varepsilon} \times B_{\varepsilon}/2} f(x,y) \, dx \, dy + 2 \int_{B_{\varepsilon} \times B_{\varepsilon}/2} f(x,y) \, dx \, dy - \int_{B_{\varepsilon} \times B_{\varepsilon}} f(x,y) \, dx \, dy
=: 2T_1 + 2T_2 - T_3.
$$

**Step 2.** By assumption, $G(x) = (G(x) \cdot \frac{x}{|x|}) \frac{x}{|x|}$ and $G(y) = 0$ for $y \in B_{R/2}$, so

$$
T_1 = \int_{B_{\varepsilon}} (G(x) \cdot \frac{x}{|x|}) \left( \int_{B_{\varepsilon}/2} \frac{x \cdot (x-y)^\perp}{|x||y-x|^2} \omega_\varepsilon(y) \, dy \right) \omega_\varepsilon(x) \, dx.
$$

Lemma 13 implies that for every $x$ such that $|x| \geq R$,

$$
\left| \int_{B_{(R/2)}} \frac{x \cdot (x-y)^\perp}{|x||y-x|^2} \omega_\varepsilon(y) \, dy \right| \leq \frac{C}{(|x| - R/2)^2 (R/2)} I(\omega_\varepsilon) \leq \frac{C}{R^3} I(\omega_\varepsilon).
$$

It follows that

$$
|T_1| \leq \frac{C\|G\|_\infty m(R)}{R^3} I(\omega_\varepsilon).
$$

**Step 3.** We now claim that

$$
\int_{B_{\varepsilon} \times B_{\varepsilon}/2} |f(x,y)| \, dx \, dy \leq \frac{C\|
abla G\|_\infty m(R)}{R^2} I(\omega_\varepsilon)
$$

and hence

$$
|T_2| + |T_3| \leq \frac{C\|
abla G\|_\infty m(R)}{R^2} I(\omega_\varepsilon).
$$

To prove this, note that by the Mean Value Theorem,

$$
|f(x,y)| \leq \frac{1}{2} \|
abla G\|_\infty \omega_\varepsilon(x) \omega_\varepsilon(y).
$$

Thus, since $1 \leq \frac{|y|^2}{(R/2)^2} = 4 \frac{|y|^2}{R^2}$ for $y \in B_{\varepsilon}/2$,

$$
\int_{B_{\varepsilon} \times B_{\varepsilon}/2} |f(x,y)| \, dx \, dy \leq \frac{2\|
abla G\|_\infty}{R^2} \int_{B_{\varepsilon}} \omega_\varepsilon(x) \, dx \int_{\mathbb{R}^2} |y|^2 \omega_\varepsilon(y) \, dy
$$

which implies the claim. □
and define

\[ \mu_t(R, h) := \frac{1}{\Omega} \int_{R^2} \left(1 - \chi_{R,h}(x - p_\varepsilon(t))\right) \omega_\varepsilon(x, t) \, dx, \]

\[ m_t(R) = \int_{B^c_r} \omega_\varepsilon(x, t) \, dx. \]

Then

\[ \frac{d}{dt} \mu_t(R, h) \leq C \left( \frac{RL}{h} + \sqrt{I_\varepsilon(t)} + \frac{I_\varepsilon(t)}{h^2R^2} \right) m_t(R) \]

**Proof.** For simplicity, in the proof we will write \( \chi \) as an abbreviation for \( \chi_{R,h} \).

We compute

\[ \frac{d}{dt} \mu_t(R, h) = \frac{1}{\Omega} \int_{R^2} \nabla \chi(x - p_\varepsilon(t)) \cdot \left[ \frac{dp_\varepsilon(t) - u(x, t) - F(x, t)}{\Omega} \right] \omega_\varepsilon(x, t) \, dx \]

\[ = T_1 + T_2, \]

where, using (117) and the Biot-Savart law,

\[ T_1 := \frac{1}{\Omega} \int_{R^2} \nabla \chi(x - p_\varepsilon(t)) \cdot \left[ \int_{R^2} F(y, t) \frac{\omega_\varepsilon(y, t)}{\Omega} \, dy - F(x, t) \right] \, dx \]

\[ T_2 := \frac{1}{\Omega} \int_{R^2 \times R^2} \nabla \chi(x - p_\varepsilon(t)) \cdot \frac{(x - y) + I_\varepsilon(t)}{\Omega} \omega_\varepsilon(x, t) \, dx \, dy \]

In estimating the two terms above, we may regard \( t \) as fixed, so we will often suppress the argument \( t \). With this notation, and observing that \( \nabla \chi \) is supported in the annulus \( A := \{x : R \leq |x - p_\varepsilon| \leq R + h\} \), we have

\[ |T_1| \leq \frac{C}{h} \int_A \left| F(x) - \int_{R^2} F(y) \frac{\omega_\varepsilon(y)}{\Omega} \, dy \right| \omega_\varepsilon(x) \, dx. \]

Since \( R \leq |x - p_\varepsilon| \leq R + h \leq 2R \) in \( A \), it follows from Lemma 14 that

\[ |T_1| \leq \frac{C}{h} (R \varepsilon + \sqrt{I_\varepsilon(t)}) m_t(R) \]

Next, Lemma 15 and properties of \( \chi \) immediately that

\[ |T_2| \leq \frac{C}{h^2R^2} m_t(R) I_\varepsilon(t), \]

and the conclusion of the lemma follows by combining these two estimates.

\[ \square \]

We now present the

**Proof of Theorem 6.**

**Step 1.** Fix \( T > 0 \). We will write \( \mu_t(R) := \mu_t(R, R) \). It is clear from the definitions that

\[ \mu_t(R) \leq m_t(R) \leq \mu_t(R/2) \quad \text{for every } t, R > 0. \]

Thus Lemma 16 implies that

\[ \mu_t(R) \leq \mu_0(R) + \kappa(R) \int_0^t \mu_{4z}(R/2) \, dt \]

for

\[ \kappa(R) = C(L + \sqrt{I_\varepsilon(t)} + \frac{\varepsilon(t)}{R^4}) \leq C(L + \varepsilon + \frac{\varepsilon^2}{R^4}) \]
We claim that almost every for every $\mu_0(R) = 0$. Thus, for any $k \in \mathbb{N}$,

$$\mu_t(2^k \sqrt{\varepsilon}) \leq C \int_0^t \mu_{t_1}(2^{k-1} \sqrt{\varepsilon}) \, dt_1.$$  

Iterating $k$ times,

$$\mu_t(2^k \sqrt{\varepsilon}) \leq C \int_0^t \cdots \int_0^{t_{k-1}} \mu_{t_k}(\sqrt{\varepsilon}) \, dt_k \cdots dt_1.$$  

Since $\mu_t(\varepsilon) \leq |\Omega|$ for all $t$ and $r$, \[\mu_t(2^k \sqrt{\varepsilon}) \leq C^k |\Omega| \int_0^t \cdots \int_0^{t_{k-1}} dt_k \cdots dt_1 = C^k |\Omega| \frac{t^k}{k!} \leq |\Omega| C^k T^{k+1/2} \varepsilon \]

where we have used Stirling's formula. Recall that the statement of the theorem involves a ball of radius $\varepsilon^a$ for some $a < 1/2$. We fix $b \in (a, 1/2)$ and we choose $k = k(\varepsilon)$ so that $\frac{1}{2} \varepsilon^b \leq 2^k \sqrt{\varepsilon} < \varepsilon^b$, leading to

$$k = \left\lfloor \frac{1}{2} b \log \varepsilon - 1 \right\rfloor$$

Then for any $\gamma_1 > 0$, we can choose $\varepsilon_0 > 0$, depending (so far) on $\gamma_1, \Omega, T > 0$ small enough that

$$\mu_t(\varepsilon^b) \leq \mu_t(2^k \sqrt{\varepsilon}) \leq |\Omega| \left( C T \varepsilon \right)^k \varepsilon^{\gamma_1}$$  

for all $t \in [0, T)$.

**Step 2.** Now consider any $\alpha \in \text{supp}(\omega_0) \subset B_{\varepsilon}(p_{\varepsilon}(0))$ and let $X(t) = X(t, \alpha)$ solve

$$\frac{dX}{dt} = (u + F)(X(t, \alpha), t), \quad X(0, \alpha) = \alpha$$

where as usual

$$u(x, t) = \int_{\mathbb{R}^2} K(x - y) \omega(y, t) \, dy.$$  

Define

$$R(t) := \max \{|X(t, \alpha) - p_{\varepsilon}(t)|, 4 \varepsilon^b\}.$$  

We claim that almost every for every $t \in [0, T)$,

$$|R'(t)| \leq C \|\nabla F\|_\infty R(t) + C \varepsilon^{1/2} \leq C (\|\nabla F\|_\infty + 1) R(t) \leq CR(t)$$

(2.120)

(where as usual the meaning of “C” changes from one appearance to the next.) If $R(t) = 4 \varepsilon^b$ and $R'(t)$ exists, then $R'(t) = 0$, so we may assume that $R(t) = |x(t) - p_{\varepsilon}(t)| > 4 \varepsilon^b$. We then compute

$$R'(t) = \frac{X(t) - p_{\varepsilon}(t)}{|X(t) - p_{\varepsilon}(t)|} \cdot \left[ u(X(t), t) + F(X(t), t) - \int F(y, t) \omega(y, t) \, dy \right].$$

From Lemma [14]

$$\left| F(X(t), t) - \int F(y, t) \omega(y, t) \, dy \right| \leq CL \left( R(t) + \sqrt{1_\varepsilon(t)} \right) \leq CLR(t) + CL \varepsilon.$$  

And

$$X(t) - p_{\varepsilon}(t) \cdot u(X(t), t) = \int_{\mathbb{R}^2} \frac{X(t) - p_{\varepsilon}(t)}{|X(t) - p_{\varepsilon}(t)|} \cdot K(x - y) \omega_{\varepsilon}(y, t) \, dy.$$
We split the integral into two pieces. Let $B(t) := B_{R(t)/2}(p(t)$ From Lemma 13

\begin{equation}
(122) \quad \left| \int_{B(t)} \frac{X(t) - p_x(t)}{|X(t) - p_x(t)|} \cdot K(x - y) \omega_x(y, t) \, dy \right| \leq \frac{C}{R(t)^3} I_\varepsilon(t) \leq C \varepsilon^{2-3b}.
\end{equation}

Next, we want to apply Lemma 12 to $\omega = 1_{B(t)} \cdot \omega_\varepsilon(\cdot, t)$. Note that
\[ \|\omega\|_p \leq \|\omega_\varepsilon(t)\|_p = \|\omega_\varepsilon^0\|_p \leq \varepsilon^{-\gamma}. \]

Also, it follows from (119) that
\[ \|\omega\|_1 = m_t \left( \frac{1}{2} R(t) \right) \leq m_t (2\varepsilon^b) \leq \mu_t (\varepsilon^b) \leq \varepsilon^{\gamma_1}. \]

Thus Lemma 12 implies that
\begin{equation}
(123) \quad \left| \int_{B(t)} \frac{X(t) - p_x(t)}{|X(t) - p_x(t)|} \cdot K(x - y) \omega_x(y, t) \, dy \right| \leq \int_{\mathbb{R}^2} \frac{\omega(y)}{|x - y|} \, dy
\end{equation}

provided we fix $\gamma_1$ in (119) to be sufficiently large. Now (120) follows from (121), (122), and (123), after possibly decreasing $\varepsilon_0$ so that $C\|F\|_\infty \varepsilon^{2-2b} \leq \varepsilon^{1/2}$.

Step 3. Since $\alpha \in \text{supp}(\omega_\varepsilon^0) \subset B(\varepsilon, p_x(0))$, it is clear that $R(0) = 2\varepsilon^b$. Then Grönwall’s inequality implies that $R(t) \leq \exp(C\|\nabla F\|_{\infty} + 1) R(0)$ and hence, after decreasing $\varepsilon_0$ as necessary, $R(t) \leq \varepsilon^a$ for $0 < t \leq T$.

This says that if $x^0 \in \text{supp}(\omega_\varepsilon^0)$, then $x^0(t) \in B(\varepsilon^a, p_x(t))$ for every $t \in [0, T]$. Since the vorticity vorticity transport formula implies that
\[ \text{supp}(\omega_\varepsilon(t)) = \{X(\alpha, t) : \alpha \in \text{supp}(\omega_\varepsilon^0)\}, \]
this completes the proof of the theorem.

2.3. Proof of Theorem \[7\]. Recall that the statement of the theorem assumes that trajectories of the point vortex ODEs (103) do not collide on a time interval $[0, T^*)$. We fix any $T < T^*$, and we must show that there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$, then conclusions (105) hold for $t \in [0, T]$.

Let $u_\varepsilon$ be the solution from the statement of the theorem. We have already discussed the starting point of the proof, which is to rewrite the Euler equations for $u_\varepsilon$ as a system
\begin{equation}
(124) \quad \partial_t \omega_\varepsilon + (u_\varepsilon + F_\varepsilon) \cdot \nabla \omega_\varepsilon = 0, \quad u_\varepsilon = \mathcal{L} \omega_\varepsilon.
\end{equation}

where $u_\varepsilon$ and $\omega_\varepsilon$ are defined in (106), (107), (108) and
\[ F_\varepsilon := \sum_{\ell \neq \varepsilon} u_\ell. \]

We remark that the definitions imply that $\int_{\mathbb{R}^2} \omega_\varepsilon(t) \, dx = \int_{\mathbb{R}^2} \omega_\varepsilon(0) \, dx = \gamma_m$ for all $m$ and $t$. This fact will be used without further mention below. We also recall that
\[ \omega_\varepsilon = \sum_{m=1}^M \omega_{\varepsilon, m} \quad \text{and} \quad u_\varepsilon = \sum_{m=1}^M u_{\varepsilon, m}. \]

We want to apply Theorems 5 and 6. The idea will be to define a time $T_\varepsilon \leq T$ such that the hypotheses of these results are satisfied for $0 \leq t \leq T_\varepsilon$, then to prove that $T_\varepsilon \geq T$ (hence $= T$) if $\varepsilon$ is sufficiently small.
Having fixed $T < T^*$ above, define
\[
\delta := \min \{ |p_m(t) - p_\ell(t)| : 0 \leq t \leq T, m \neq \ell \} > 0
\]
and
\[
T_\varepsilon := \sup \left\{ t \in [0, T] : \text{supp}(\omega_{\varepsilon, m}(s)) \subset B_{\delta/4}(p_m(s)) \text{ for } m = 1, \ldots, M \text{ and } 0 \leq s \leq t \right\}.
\]
Since
\[
|\nabla F_{\varepsilon, m}(x)| \leq \sum_{\ell \neq m} \int_{\text{supp}(\omega_{\varepsilon, \ell})} |\nabla K(x - y)| \, |\omega_{\varepsilon}(y)| \, dy \leq C \sum_{\ell \neq m} \int_{\text{supp}(\omega_{\varepsilon, \ell})} \frac{|\omega_{\varepsilon}(y)|}{|x - y|^2} \, dy
\]
it is clear that
\[
\text{if } 0 \leq t \leq T_\varepsilon \text{ and } x \in B_{\delta/4}(p_m(t)), \quad \text{then } \quad |\nabla F_{\varepsilon, m}(x)| \leq L := C \frac{\|\omega_{\varepsilon}\|_{L^1}}{\delta^2}.
\]
(No\-te that $\|\omega_{\varepsilon}\|_{L^1} = \sum_{m=1}^M |\gamma_m|$, so $L$ is independent of $\varepsilon$.) We now invoke Theorems 5 and 6 for $t \in [0, T_\varepsilon]$ with this choice of $L$, and with $a$ replaced by $b$ such that $a < b < 1/2$. To write the conclusions, we introduce the notation
\[
p_{\varepsilon, m}(t) = \frac{1}{\gamma_m} \int_{\mathbb{R}^2} x \omega_{\varepsilon, m}(x, t) \, dx,
\]
\[I_{\varepsilon, m}(t) := \frac{1}{2\gamma_m} \int_{\mathbb{R}^2} |x - p_{\varepsilon, m}(t)|^2 \omega_{\varepsilon, m}(x, t) \, dx,
\]
and we write $P_{\varepsilon, m}(t)$ to denote the solution of the ODEs
\[
\frac{d}{dt} p_{\varepsilon, m}(t) = F_{\varepsilon, m}(P_{\varepsilon, m}(t), t), \quad P_{\varepsilon, m}(0) = p_{\varepsilon, m}(0).
\]
Then the theorems imply that for $m = 1, \ldots, M$ and $0 \leq t \leq T_\varepsilon$,
\[
I_{\varepsilon, m}(t) \leq C \varepsilon^2, \quad |p_{\varepsilon, m}(t) - P_{\varepsilon, m}(t)| \leq C \varepsilon, \quad \text{supp}(\omega_{\varepsilon, m}(t)) \subset B_{\varepsilon^2}(p_{\varepsilon, m}(t))
\]
whenever $\varepsilon \leq \varepsilon_0$, and thus that
\[
\text{supp}(\omega_{\varepsilon, m}(t)) \subset B_{\varepsilon^2 + C \varepsilon} + \varepsilon \text{supp}(P_{\varepsilon, m}(t)) \subset B_{2\varepsilon^2} + \varepsilon \text{supp}(P_{\varepsilon, m}(t)) \quad \text{if } \varepsilon \text{ is small enough.}
\]
Next, we set
\[
F_m(x, t) := \sum_{\ell \neq m} \gamma_\ell K(x - p_\ell(t)),
\]
that is, the vector field on the right-hand side of the point vortex ODEs.

Then for $x \in B_{\delta/4}(p_m(t))$,
\[
|F_{\varepsilon, m}(x, t) - F_m(x, t)| = \sum_{\ell \neq m} \left| \int_{\text{supp}(\omega_{\varepsilon, \ell}(t))} [K(x - y) - K(x - p_\ell(t))] \omega_{\varepsilon, \ell}(y, t) \, dy \right|
\leq \sum_{\ell \neq m} \int_{\text{supp}(\omega_{\varepsilon, \ell}(t))} |K(x - y) - K(x - p_\ell(t))| |\omega_{\varepsilon, \ell}(y, t)| \, dy
\]
For \( x \in B_{\delta/4}(p_m(t)) \) and \( y, p_\ell \in \text{supp}(\omega_{\epsilon,m}(t)) \) with \(|p_m(t) - p_\ell(t)| \geq \delta\), by the Mean Value Theorem there exists \( \theta \in (0,1) \) such that
\[
|K(x - y) - K(x - p_\ell(t))| \leq |y - p_\ell(t)| |\nabla K(x - \theta y - (1 - \theta)p_\ell(t)|
\]
\[
\leq C \frac{\delta^2}{\epsilon^2} |y - p_\ell(t)|
\]
\[
\leq C \frac{\delta^2}{\epsilon^2} (|p_{\epsilon,\ell}(t) - p_\ell(t)| + 2\epsilon^b).
\]

So for \( x \in B_{\delta/4}(p_m(t)) \),
\[
|F_{\epsilon,m}(x,t) - F_m(x,t)| \leq C \sum_{\ell \neq m} (|p_{\epsilon,\ell}(t) - p_\ell(t)| + 2\epsilon^b)
\]

Then comparing the ODEs (103) for \( p_m \) and (117) for \( p_{\epsilon,m} \), we see that
\[
\frac{d}{dt} \left( \sum_{m=1}^{M} |p_{\epsilon,m}(t) - p_m(t)| \right) \leq \sum_{m=1}^{M} \frac{d}{dt} (|p_{\epsilon,m}(t) - p_m(t)|)
\]
\[
\leq \sum_{m=1}^{M} |F_{\epsilon,m}(p_{\epsilon,m}(t),t) - F_m(p_m(t),t)|
\]
\[
\leq C \sum_{\ell=1}^{M} (|p_{\epsilon,\ell}(t) - p_\ell(t)| + 2\epsilon^b).
\]

Also, \( P_{\epsilon,m}(0) = p_m(0) \) for all \( m \). It follows that
\[
\sum_{m=1}^{M} |P_{\epsilon,m}(t) - p_m(t)| \leq e^{C\epsilon}2\epsilon^b \quad \text{for} \quad 0 \leq t \leq T_\epsilon
\]

and thus in view of (127),
\[
\text{supp}(\omega_{\epsilon,m}(t)) \subset B_{C\epsilon^b}(p_m(t)) \quad \text{for all} \quad m, \quad 0 \leq t \leq T_\epsilon.
\]

If \( \epsilon \) is sufficiently small, then \( C\epsilon^b \leq \epsilon^a \), and when this is the case it follows that \( T_\epsilon \geq T \).

2.4. Vortex dynamics on a bounded domain. It is easy to modify the above argument to address the Euler equations on an open, connected and simply connected \( \mathbb{D} \subset \mathbb{R}^2 \) with smooth boundary. Then the equations are the same as always, e.g., in the velocity-pressure formulation
\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= 0, \\
\nabla \cdot u &= 0
\end{align*}
\]
\[
\text{in} \ \mathbb{D} \times (0, T).
\]

supplemented by the boundary conditions
\[
\text{EulerBC2} \quad (129) \quad u \cdot \mathbf{v} = 0 \quad \text{on} \ \partial \mathbb{D} \times (0, T).
\]

where \( \mathbf{v} \) is the outer unit normal.

We briefly discuss the modifications necessary to address this case.

First, the boundary conditions play a role in the relationship between the velocity \( u \) and the vorticity \( \omega \), now given by the system of equations
\[
\begin{align*}
\nabla \cdot u &= 0, \\
\nabla^\perp \cdot u &= \omega
\end{align*}
\]
\[
\text{in} \ \mathbb{D}, \quad u \cdot \mathbf{v} = 0 \quad \text{on} \ \partial \mathbb{D}.
\]
By analogy with the Biot-Savart law, we are interested in the problem of reconstructing \( \mathbf{u} \) from a knowledge of \( \omega \). We recall from vector calculus that our assumption that \( D \) is simply-connected implies that given \( \omega \), a solution \( \mathbf{u} \) of (130), if it exists, must be unique. To see that a solution exists, we will write one down. We assume for convenience that \( \omega \in L^\infty(D) \), although weaker assumptions would suffice.

Given \( \omega \), we can reconstruct \( \mathbf{u} \) as follows. First, we define the stream function \( \psi \) associated to vorticity \( \omega \) to be the function \( \psi \) such that

\[
-\Delta \psi(x) = \omega(x) \quad \text{in} \quad \Omega
\]

and \( \psi = 0 \) on \( \partial \Omega \). One can verify that

\[
\psi(x) = \int_{\Omega} G(x, y) \omega(y) \, dy
\]

where

\[
G(x, y) = \frac{1}{2\pi} \log \left( \frac{1}{|x - y|} \right) + H(x, y)
\]

and \( H \) is the unique function such that for every \( y \in \Omega \),

\[
x \mapsto H(x, y)
\]

is harmonic, and \( K(x, y) = 0 \) for \( x \in \partial \Omega \).

Thus, for every \( y \in \Omega \), \( x \mapsto H(x, y) \) solves

\[
-\Delta x H(x, y) = 0, \quad H(x, y) = -\frac{1}{2\pi} \log \left( \frac{1}{|x - y|} \right) \quad \text{for} \quad x \in \partial \Omega.
\]

It is a standard fact from basic PDE classes that the problem (133) has a unique solution.

Then we define \( \mathbf{u}(x) = -\nabla \perp \psi(x) \). In other words,

\[
\mathbf{u}(x) = \mathcal{K} \omega(x) = \int_{\Omega} K(x, y) \omega(y) \, dy, \quad K(x, y) = \frac{(x - y)^{\perp}}{2\pi|x - y|^2} - \nabla \perp^x H(x, y).
\]

Clearly, if \( \omega = \sum_{m=1}^{M} \gamma_m \delta_{p_m} \), then

\[
\mathcal{K} \omega = \sum_{m=1}^{M} \nabla \perp G(x, p_m) = \sum_{m=1}^{M} \frac{(x - y)^{\perp}}{2\pi|x - y|^2} - \nabla \perp^x H(x, y).
\]

Based on this, what law of motion would we predict for a collection of point vortices? At the position \( p_m \) of the \( m \)th vortex, the velocity field consists of

- The contributions \( \sum_{\ell \neq m} \gamma_{\ell} \nabla \perp G(x, p_\ell) \) from the other vortices.
- The divergent part of the velocity field that the vortex itself generates, \( \frac{(x - p_m)^{\perp}}{2\pi|x - p_m|^2} \). This is singular at \( p_m \) but also purely rotational, so we may suppose that it does not have any impact on vortex motion at all.
- The regular part \( -\nabla \perp^x H(x, p_m) \) of the velocity field that the vortex itself generates (via its interaction with \( \partial \Omega \)). There is no reason to suppose that this will have no affect on the vortex motion.

So adding up the contributions that we think are relevant and ignoring the one that we think (hope) is not, and applying the same reasoning to every vortex, we might
predict that the law of motion should be given by

\[
\frac{dp_m}{dt} = -\gamma_m \nabla_x^\perp H(p_m, p_m) - \sum_{\ell \neq m} \gamma_\ell \nabla_x^\perp G(p_m, p_\ell)
\]

for every \( m \) with initial data \( \gamma_m, \ldots, \gamma_M \in \mathbb{R} \) and points \( p^0_1, \ldots, p^0_M \in \mathcal{D} \), and let \( p(t) = (p_1(t), \ldots, p_M(t)) : [0, T^*) \to \mathcal{D}^M \) solve (135) with initial data \( p_m(0) = p^0_m \).

Assume also that \( p_m(t) \neq p_\ell(t) \) for all \( t \in [0, T^*) \) and \( m \neq \ell \).

**THEOREM 7.** Let \( u_\varepsilon \in L^\infty(\mathcal{D} \times [0, \infty), \mathbb{R}^2) \) be the unique (Yudovich) solution to the Euler equations (128) with boundary conditions (129) for initial data \( u^0_\varepsilon \) such that the vorticity \( \omega^0_\varepsilon = \nabla^\perp \cdot u^\varepsilon \) belongs to \( L^\infty(\mathcal{D}) \) and satisfies (137)

\[
\supp(\omega^0_\varepsilon) \subset \bigcup_{m=1}^M B_\varepsilon(p^0_m), \quad \int_{B_\varepsilon(p^0_m)} \omega^0_\varepsilon \, dx = \gamma_m, \quad \gamma_m \omega^0_\varepsilon(x) > 0 \text{ in } B_\varepsilon(p^0_m).
\]

Assume also that there exists some \( \alpha > 0 \) and \( p > 2 \) such that

\[
\|\omega^0_\varepsilon\|_{L^p(\mathcal{D})} \leq \varepsilon^{-\gamma}.
\]

Then for any \( T < T^* \) and any \( \alpha < 1/2 \), there exist \( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon \leq \varepsilon_0 \), (138)

\[
\supp(\omega_\varepsilon(t)) \subset \bigcup_{m=1}^M B_\varepsilon(p_m(t)), \quad \int_{B_\varepsilon(p_m(t))} \omega_\varepsilon(x, t) \, dx = \gamma_m.
\]

for every \( t \in [0, T] \), where \( p_m(t) \) solves (135). Furthermore, \( \varepsilon_0 \) depends only on the initial data for (135), \( T, p, \gamma, a \).

The proof is extremely similar to that of Theorem 4. The starting point is rewriting the Euler equations as a system

\[
\partial_t \omega_{\varepsilon, m} + (u_{\varepsilon, m} + F_{\varepsilon, m}) \cdot \nabla \omega_{\varepsilon, m} = 0,
\]

where \( \omega_{\varepsilon, m} \) is defined exactly as before, see (108), and

\[
u_{\varepsilon, m}(x, t) = \int_{\supp(\omega_{\varepsilon, m})} \frac{(x - y)^\perp}{2\pi|x - y|^2} \omega_{\varepsilon, m}(y, t) \, dy
\]

(exactly as in Theorems 5 and 6) and

\[
F_{\varepsilon, m}(x, t) = -\sum_{\ell \neq m} \int_{\mathbb{R}^2} \nabla^\perp G(x, y) \omega_{\varepsilon, \ell}(y, t) \, dy - \int_{\mathbb{R}^2} \nabla^\perp H(x, y) \omega_{\varepsilon, m}(y, t) \, dy.
\]

Then we can invoke Theorems 5 and 6) exactly as before. The computations from the earlier proof that involve \( F_{\varepsilon, m} \) and \( F_m \) have to be modified in rather natural ways, and everything else is pretty much the same. We omit the details.
CHAPTER 3

Vortex rings: the axisymmetric Euler equations without swirl

We now consider fluid flows (inviscid, incompressible) in $\mathbb{R}^3$ with cylindrical symmetry. This is the natural setting for studying the phenomenon of leapfrogging vortex rings, pictured below.

For this class of problems, it is natural to use cylindrical coordinates $(r, \theta, z)$, related as usual to Cartesian coordinates by

$$(x_1, x_2, x_3) = (r \cos \theta, r \sin \theta, z).$$

In this section, we will use arrows to designate vector fields in $\mathbb{R}^3$. Thus, we will write $\vec{e}_j, j = 1, 2, 3$ to denote the standard Cartesian unit vectors, and $\vec{e}_r, \vec{e}_\theta, \vec{e}_z$ to represent the unit vectors associated to cylindrical coordinates:

\[
\begin{align*}
\vec{e}_r &= \frac{x_1}{r} \vec{e}_1 + \frac{x_2}{r} \vec{e}_2 = \cos \theta \vec{e}_1 + \sin \theta \vec{e}_2 \\
\vec{e}_\theta &= -\frac{x_2}{r} \vec{e}_1 + \frac{x_1}{r} \vec{e}_2 = -\sin \theta \vec{e}_1 + \cos \theta \vec{e}_2 \\
\vec{e}_z &= \vec{e}_3.
\end{align*}
\]

We will frequently use the notation

$$\mathbb{H} := \{(r, z) \in \mathbb{R}^2 : r > 0\}.$$

For functions on $\mathbb{H}$, we will write

$$\nabla := (\partial_r, \partial_z), \quad \nabla^\perp := (-\partial_z, \partial_r).$$
We will also continue to use $\nabla$ to mean $(\partial_1, \partial_2, \partial_3)$ in $\mathbb{R}^3$ with Cartesian coordinates. This should not result in any ambiguity.

An axisymmetric flow without swirl \cite{AWoS} is one for which the velocity field has the form

\[
\vec{u}(x) = u^r(r, z)\vec{e}_r + u^\theta(r, z)\vec{e}_\theta + u^z(r, z)\vec{e}_z
\]

It is not hard to check that $\vec{u} : \mathbb{R}^3 \times [0, T) \to \mathbb{R}^3$ has the above form for every $t$, then $\vec{u}$ solves the 3d Euler equations

\[
\begin{align*}
\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + \nabla p &= 0 \\
\nabla \cdot \vec{u} &= 0
\end{align*}
\]

if and only if $u = (u^r, u^z) : \mathbb{H} \times [0, T) \to \mathbb{R}^2$ solves the system

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= 0 \\
\frac{1}{r} \nabla \cdot (ru) &= 0.
\end{align*}
\]

where, lest there be any ambiguity,

\[
u \cdot \nabla = u^r \partial_r + u^z \partial_z, \quad \nabla p = (\partial_r p, \partial_z p), \quad \frac{1}{r} \nabla \cdot (ru) = \frac{1}{r} u^r + \partial_r u^r + \partial_z u^z.
\]

We add the boundary condition $u^r = 0$ on $\partial H$, which we write as

\[
u \cdot \nu = 0 \quad \text{on} \quad \partial D.
\]

If $u$ is continuous at $\partial H$, then in view of \eqref{form}, this is equivalent to continuity of $\vec{u}$ on $\{(r \sin \theta, r \cos \theta, z) \in \mathbb{R}^3 : r = 0\}$.

As usual, we define the vorticity

\[
\omega := \nabla \times \vec{u} = \nabla \times \vec{u} = \nabla (r, z) \vec{e}_\theta
\]

and we remark that if $\vec{u}$ and $u$ are related as in \eqref{inter}, then

\[
\vec{\omega} := \nabla \times \vec{u} = \omega(r, z)\vec{e}_\theta
\]

as is checked by a short calculation.

The study of concentrated vortex rings, for us, will mean the study of solutions of \eqref{AWoS1} with concentrated vorticity, approximating one or more point vortices, in the $(r, z)$-halfplane $\mathbb{H}$. These could be either solutions that evolve by a rigid motion, or solutions of equation \eqref{AWoS1} for initial data similar to that considered in Section 2 above.

1. Basic properties

The Euler equations for axisymmetric flows without swirl \eqref{AWoS1} can be viewed as a special case of a more general equation

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= 0 \\
\frac{1}{b} \nabla \cdot (bu) &= 0
\end{align*}
\]

where $\mathcal{D}$ is an open subset of $\mathbb{R}^2$ and $b : \mathcal{D} \to (0, \infty)$ is a given positive function. Versions of this equation arise in a couple of ways:

\footnote{If $\vec{u}$ has the form velocity $\vec{u}$ has the form $u^r(r, z)\vec{e}_r + u^0(r, z)\vec{e}_r + u^z(r, z)\vec{e}_z$, then is is said to be axisymmetric with swirl.}
• As we have seen, (144) with $D = H$ and $b(r, z) = r$ arises as a symmetry reduction of 3d Euler. Similarly, there are various symmetry reductions of the incompressible Euler equations in $\mathbb{R}^N$ for $N \geq 4$ that give rise to equations of the form (144), for example with $D = (0, \infty)^2$, and $b(x_1, x_2) = x_1^m x_2^\ell$, for integers $m, \ell \geq 1$.

• If $D$ represents the surface of a body of an incompressible, inviscid fluid (eg, an idealization of water) in $\mathbb{R}^2$ whose depth at $x \in D$ is given by $b(x)$, and if the depth is much smaller than the horizontal dimensions of $D$, then (144) models the horizontal velocity of the fluid in $D$, averaged vertically. With this interpretation in mind, (144) are sometimes called the Lake Equations.2

1.1. conserved quantities. We start by discussing some general properties of (144), before specializing to the case relevant for vortex rings. First, by applying $\nabla^\perp \cdot$ to (144), we obtain the equation

$$\frac{D}{Dt} \omega + \omega \nabla \cdot u = 0$$

where $\frac{D}{Dt} = \partial_t + u \cdot \nabla$ as usual. From this and the incompressibility condition $\nabla \cdot (bu) = 0$, one sees that

$$\frac{D}{Dt} (\omega b) = 0.$$  

It follows that if we define the particle trajectory map as usual by

$$\frac{d}{dt} X(\alpha, t) = u(X(\alpha, t), t), \quad X(\alpha, 0) = \alpha$$

then

$$\frac{\omega}{b} (X(\alpha, t), t) = \frac{\omega}{b}(\alpha, 0).$$

This is the Vorticity Transport Formula for (144). We can also rewrite (146) as

$$\partial_t \omega + \nabla \cdot (u \omega) = 0$$

from which it follows that

$$\frac{d}{dt} \int_D \frac{1}{2} b |u|^2 \, dx = 0.$$  

This says that the total circulation is conserved.

It is evident that for a solution of (144),

$$\frac{1}{2} \frac{d}{dt} \int_D (b|u|^2) = -bu \cdot (u \cdot \nabla)u - bu \cdot \nabla p = -\nabla \cdot \left[ bu \left( \frac{1}{2} |u|^2 + p \right) \right].$$

This in particular implies that

$$\frac{d}{dt} \int_D \frac{1}{2} b |u|^2 \, dx = 0$$

if (143) holds where, in the case of an unbounded domain, we require suitable decay at infinity. This is conservation of energy for (144).

2Arguably, however, this name is inappropriate for the versions of (144) arising from symmetry reductions of high-dimensional Euler.
Finally, a different form of the (144) is
\[ \partial_t u + \frac{1}{b} \nabla \cdot (bu \otimes u) + \nabla p = 0. \]
Applying \( \nabla \perp \cdot \) to this, multiplying by a test function \( \varphi = \varphi(x) \), and integrating, we find after several integrations by parts that
\[ \partial_t \int_D \varphi \omega \, dx = \int_D \nabla \left( \frac{1}{b} \nabla \perp \varphi \right) : bu \otimes u \, dx. \]
Thus \( \int \varphi \omega \) is conserved for any \( \varphi \) such that \( \nabla \left( \frac{1}{b} \nabla \perp \varphi \right) = 0 \). This amounts to 4 equations:
\[ \partial_i \left( \frac{1}{b} \partial_j \varphi \right) = 0 \quad \text{for} \quad i, j = 1, 2. \]
In general one does not expect to find solutions, but note that if \( b(r, z) = f(r) \), then \( \varphi(r, z) = \int_0^r f(s) \, ds \) is a solution. In particular
\[ (151) \]
for axisymmetric Euler without swirl, \( \int_H r^2 \omega(r, z, t) \, dr \, dz \) is conserved.
Thus, a major difference between axisymmetric Euler without swirl and the general equation (144) is the presence of one additional conserved quantity for the former.

The other major difference is that we can find an explicit formula for the Biot-Savart Law in this setting. In general, for (144), we can attempt to represent \( u \) in the form
\[ u = - \frac{\nabla \perp \psi}{b}, \quad \text{where} \quad \left\{ \begin{array}{l}
- \nabla \cdot \left( \frac{\nabla \psi}{b} \right) = \omega \text{ in } \mathcal{D} \\
\psi = 0 \text{ on } \partial \mathcal{D}.
\end{array} \right. \]
To the extent that one can find a useful formula for \( \psi \) (which solves a second-order elliptic equation), one can obtain a useful formula for \( u \). For a general weight function \( b \) in (144) there is no explicit formula\(^3\) for \( \psi \). However, there classical formulas are available in the case of (142). These are the subject of the next section.

1.2. The Biot-Savart Law. Assume that \( \omega : \mathbb{H} \to \mathbb{R} \) is bounded, with rapid decay. Here we derive formulas for solutions of the problems
\[ (152) \]
\[ - \nabla \cdot \left( \frac{\nabla \psi}{r} \right) = \omega \text{ on } \mathbb{H}, \quad \psi = 0 \text{ on } \partial \mathbb{H}. \]
and
\[ (153) \]
\[ \nabla \perp \cdot u = \omega \quad \text{in } \mathbb{H}, \quad u \cdot \nu = 0 \text{ on } \partial \mathbb{H}. \]
The formula for \( \psi \) is useful both in deriving the formula for \( u \), and in expressing the kinetic energy of a solution of (142).

**Lemma 17.** For \( \omega \) as above, a solution \( \psi \) of (152) is given by
\[ (154) \]
\[ \psi(r, z) = \int_{\mathbb{H}} G_{\text{axs}}(r, z, r', z') \omega(r', z') \, dr' \, dz', \]
where
\[ (155) \]
\[ G_{\text{axs}}(r, z, r', z') := \int_{-\pi}^{\pi} \frac{\cos \alpha \, d\alpha}{4\pi((z - z')^2 + (r - r')^2 + 2rr'(1 - \cos \alpha))^{1/2}}. \]

\(^3\)However, it is sometimes possible to give useful approximate approximations, as in recent work of Dekeyser and van Schaftingen. We will not pursue this here however.
A solution \( \mathbf{u} = (u^r, u^z) \) of (153) is given by

\[
\mathbf{u} \text{.kernel1} \quad (156) \quad [u^r \mathbf{e}_r + u^z \mathbf{e}_z](r, z) = \int_{\mathbb{H}} K_{\text{axs}}(r, z, r', z') \omega(r', z') r' \, dr' \, dz'
\]

where

\[
\mathbf{Kcyl.def} \quad (157) \quad K_{\text{axs}}(r, z, r', z') := \int_{\pi}^{\pi} \frac{[(z - z') \cos \alpha \mathbf{e}_r + (r' - r \cos \alpha) \mathbf{e}_z]}{4\pi\left((z - z')^2 + (r - r')^2 + 2rr'(1 - \cos \alpha)\right)^{3/2}} \, d\alpha.
\]

Finally,

\[
\mathbf{KE.cyl} \quad (158) \quad \pi \int_{\mathbb{H}} |\mathbf{u}|^2 r \, dr \, dz = \pi \int_{\mathbb{H}} G_{\text{axs}}(r, z, r', z') \omega(r, z) \omega(r', z') r' \, dr' \, dr' \, dz'.
\]

We will continue to use the notation \( \mathbf{u} = \mathcal{K}\omega \) when \( \mathbf{u} \) and \( \omega \) are related as in (156), rather than writing for example \( \mathcal{K}_{\text{axs}} \). Below we present approximate formulas for \( G_{\text{axs}} \) and \( K_{\text{axs}} \), in Lemmas 18 and 19 respectively, that exhibit some similarities with the corresponding kernels on \( \mathbb{R}^2 \).

**Proof.** Given \( \omega : \mathbb{H} \to \mathbb{R} \), we define \( \bar{\omega} : \mathbb{R}^3 \to \mathbb{R}^3 \) by

\[
\bar{\omega}(x) := \omega(r, z) \mathbf{e}_\theta.
\]

It is clear that \( \nabla \cdot \bar{\omega} = 0 \). We can thus write down formulas for \( \bar{\mathbf{\psi}}, \bar{\mathbf{u}} \) such that

\[
-\Delta \bar{\mathbf{\psi}} = \omega \quad \text{and} \quad \bar{\mathbf{u}} = \nabla \times \bar{\mathbf{\psi}},
\]

so that \( \begin{cases} 
\nabla \times \bar{\mathbf{u}} = \omega \\
\nabla \cdot \bar{\mathbf{u}} = 0.
\end{cases} \)

As we will see, \( \bar{\mathbf{u}} \) and \( \bar{\mathbf{\psi}} \) have the forms

\[
\bar{\mathbf{u}}(x) = u^r(r, z) \mathbf{e}_r + u^z(r, z) \mathbf{e}_z
\]

\[
\bar{\mathbf{\psi}}(x) = \frac{\psi(r, z)}{r} \mathbf{e}_\theta,
\]

and it is straightforward to check that

\[
\bar{\mathbf{u}} = \nabla \times \bar{\mathbf{\psi}} \quad \iff \quad \mathbf{u} = \frac{\nabla \times \mathbf{\psi}}{r}
\]

\[
\mathbf{axis.wos} \quad (159) \quad \bar{\omega} = \nabla \times \bar{\mathbf{u}} \quad \iff \quad \omega = -\nabla \cdot \mathbf{u} = -\nabla \cdot \left( \frac{\nabla \mathbf{\psi}}{r} \right).
\]

Thus formulas for \( \bar{\mathbf{\psi}}, \bar{\mathbf{u}} \) yield explicit representations of solutions \( \mathbf{\psi}, \mathbf{u} \) of (152) and (153).

To carry out the details, we write \((r, \theta, z)\) and \((r', \theta', z')\) to denote cylindrical coordinates corresponding to points \( \mathbf{x} \) and \( \mathbf{x}' \) respectively. First, using the form of the \( \bar{\omega} \) and \( \mathbf{K} \) (the Biot-Savart kernel in \( \mathbb{R}^3 \)), one checks that

\[
K(\mathbf{x} - \mathbf{x}') \bar{\omega}(\mathbf{x}') = \left[ \frac{(x_3 - x_3') \mathbf{e}_r + (x_1' - x_1) \frac{x_1'}{r'} + (x_2' - x_2) \frac{x_2'}{r'}}{4\pi|\mathbf{x} - \mathbf{x}'|^3} \right] \omega(r', x_3') = \frac{(z - z') \mathbf{e}_r + (r' - r \cos(\theta - \theta')) \mathbf{e}_z}{4\pi((z - z')^2 + (r - r')^2 + 2rr'(1 - \cos(\theta - \theta')))^{3/2}} \omega(r', z').
\]
Clearly $\vec{e}_z = \vec{e}_z$. We write $\vec{e}_r = \cos(\theta' - \theta)\vec{e}_r + \sin(\theta' - \theta)\vec{e}_\theta$, we integrate, changing changing variables by setting $\alpha = \theta - \theta'$. to find that

$$\int_{\mathbb{R}^3} K(x - x') \omega(x') dx'$$

$$= \int_{\mathbb{R}^3} \left( \int_{-\pi}^{\pi} \frac{(z - z')(\cos \alpha \vec{e}_r - \sin \alpha \vec{e}_\theta) + (r' - r \cos \alpha) \vec{e}_{z'}}{4\pi r' \left( (z - z')^2 + (r - r')^2 + 2rr'(1 - \cos \alpha)^2 \right)^{1/2}} \omega(r', z') \, d\alpha \right) \, r' \, dr' \, dz'.$$

Then parity considerations imply that the term containing $\sin \alpha$ integrates to 0. We therefore deduce \((156)\), \((157)\).

Similarly, writing $\bar{\vec{e}}_0' = \cos(\theta' - \theta)\bar{\vec{e}}_0 - \sin(\theta' - \theta)\bar{\vec{e}}_r$, we see that

$$G(x - x')\bar{\omega}(x') = \frac{\cos(\theta' - \theta)\bar{\vec{e}}_0 - \sin(\theta' - \theta)\bar{\vec{e}}_r}{4\pi r' \left( (z - z')^2 + (r - r')^2 + 2rr'(1 - \cos(\theta - \theta')) \right)^{1/2}} \omega(r', z'),$$

from which we deduce \((154)\), \((155)\) by arguing as above.

Finally, since $\bar{\vec{u}} = \nabla \times \psi$, we integrate by parts to find that

$$\int_{\mathbb{R}^3} \left| \bar{\psi}(x) \right|^2 = \int_{\mathbb{R}^3} \bar{\omega}(x') \, dx'$$

$$= \int_{\mathbb{R}^3} \left( \frac{1}{4\pi |x - x'|} - \frac{1}{|x|} \right) \bar{\omega}(x') \, dx'.$$

For $|x'| \leq R$ and $|x| \geq 2R$,

$$\frac{1}{|x - x'|} - \frac{1}{|x|} \leq C \frac{1}{|x|^2}.$$

It follows that $|\bar{\psi}(x)| \leq C|x|^{-2}$ for $|x| \geq 2R$. Essentially the same considerations imply that $|\bar{\vec{u}}(x)| \leq C|x|^{-3}$ for $|x| \geq 2R$. Now for $L > 0$ let $\chi_L : \mathbb{R}^3 \to [0, 1]$ be a radial decreasing function such that

$$\chi_L(x) = 1 \text{ if } |x| \leq L, \quad \chi_L(x) = 0 \text{ if } |x| \geq 2L, \quad |\nabla \chi_L(x)| \leq \frac{C}{L}.$$

Then

$$\int_{\mathbb{R}^3} |\bar{\vec{u}}|^2 \, dx = \lim_{L \to \infty} \int_{\mathbb{R}^3} \nabla \times \psi \cdot \bar{\vec{u}} \chi_L \, dx$$

$$= \lim_{L \to \infty} \int_{\mathbb{R}^3} \chi_L \bar{\psi} \cdot \nabla \times \bar{\vec{u}} \, dx + \lim_{L \to \infty} \int_{\mathbb{R}^3} \bar{\psi} \cdot (\nabla \chi_L \times \bar{\vec{u}}) \, dx.$$

The decay of $\bar{\vec{u}}$ and $\bar{\psi}$ implies that for $L \geq 2R$,

$$\int_{\mathbb{R}^3} \bar{\psi} \cdot (\nabla \chi_L \times \bar{\vec{u}}) \, dx \leq \frac{C}{L} \int_{\{|x| \leq |x| \leq 2L\}} C|x|^{-5} \, dx \to 0 \text{ as } L \to \infty$$

and \((161)\) follows.

\square

It is sometimes useful and/or informative to rewrite the above kernels in ways that facilitate comparison with the corresponding expressions on $\mathbb{R}^2$. We first consider $G_{\text{axis}}$. 

\[\text{rewriteKE}\]
**Lemma 18.**

**F.**

\[ (162) \quad \mathbb{G}_{\text{aux}}(r, z, r', z') = \frac{\sqrt{rr'}}{2\pi} F \left( \frac{(r-r')^2 + (z-z')^2}{rr'} \right) \]

for

**F.**

\[ (163) \quad F(s) := \int_0^\pi \frac{\cos \alpha \, d\alpha}{\sqrt{s + 2(1 - \cos \alpha)}}. \]

Moreover, there exists \( C > 0 \) such that

\[ (164) \quad \log(\frac{2+\sqrt{4+s}}{\sqrt{s}}) - C\sqrt{1+s} \leq F(s) \leq \log(\frac{2+\sqrt{4+s}}{\sqrt{s}}) \quad \text{for all} \quad s \in (0, \infty). \]

In particular, \( F(s) = \frac{1}{2} \log \frac{1}{s} + O(1) \) as \( s \to 0. \)

**Remark 4.** Of course it is possible to extract more information about \( F \) by more careful analysis of the error terms. In this way one can check for example that

\[ F(s) = \frac{1}{2} \log \frac{1}{s} + \log 8 - 2 + O(s \log \frac{1}{s}) \quad \text{as} \quad s \to 0. \]

This can also be deduced by rewriting \( F \) in terms of special functions known as elliptic integrals and appealing to classical asymptotic expansions of these functions.

**Proof.** One obtains (162), (163) by simply rewriting the definition (155) of \( \mathbb{G}_{\text{aux}}. \) Then

\[
F(s) = \int_0^\pi \frac{\cos \alpha \, d\alpha}{\sqrt{s + 2(1 - \cos \alpha)}} = \int_0^\pi \frac{\cos(\alpha/2) \, d\alpha}{\sqrt{s + 4 \sin^2(\alpha/2)}} + \int_0^\pi \frac{\cos \alpha - \cos(\alpha/2)}{\sqrt{s + 4 \sin^2(\alpha/2)}} \, d\alpha
\]

The first integral is

\[
\int_0^\pi \frac{\cos(\alpha/2) \, d\alpha}{\sqrt{s + 4 \sin^2(\alpha/2)}} = \int_0^2 \frac{1}{\sqrt{t + s + t^2}} \, dt = \left[ \log(t + \sqrt{s + t^2}) \right]_0^2
\]

\[ = \log(\frac{2+\sqrt{4+s}}{\sqrt{s}}). \]

Next, by Taylor expansions one sees that

\[ \cos \alpha - \cos(\alpha/2) = -\frac{3}{8} \alpha^2 + O(\alpha^2) = -\frac{3}{2} \sin^2(\alpha/2) \cos(\alpha/2) + O(\alpha^4). \]

So

\[
\int_0^\pi \frac{\cos \alpha - \cos(\alpha/2)}{\sqrt{s + 4 \sin^2(\alpha/2)}} \, d\alpha = \frac{1}{2} \int_0^\pi \frac{\sin^2(\alpha/2) \cos(\alpha/2)}{\sqrt{s + 4 \sin^2(\alpha/2)}} \, d\alpha + \int_0^\pi \frac{O(\alpha^4)}{\sqrt{s + 4 \sin^2(\alpha/2)}} \, d\alpha
\]

And

\[
\int_0^\pi \frac{\sin^2(\alpha/2) \cos(\alpha/2)}{\sqrt{s + 4 \sin^2(\alpha/2)}} \, d\alpha = \frac{1}{4} \int_0^2 \frac{t^2}{\sqrt{s + t^2}} \, dt
\]

\[ = \frac{1}{4} \left[ \frac{1}{2} t \sqrt{s + t^2} - \frac{s}{2} \log(t + \sqrt{s + t^2}) \right]_{t=0}^2
\]

\[ = \frac{1}{4} \sqrt{s + 4} - \frac{s}{8} \log(\frac{2 + \sqrt{s + 4}}{\sqrt{s}}). \]
The second integral is clearly negative. And since \(\sin(\alpha/2) \geq c\alpha\) for \(\alpha \in (0, \pi)\) and
\[
\cos \alpha - \cos(\alpha/2) = (1 - \frac{1}{2} \alpha^2 + O(\alpha^4)) - (1 - \frac{1}{2}(\alpha/2)^2 + O(\alpha^4)) = -\frac{3}{8} \alpha^2 + O(\alpha^4)
\]
it is clear that there exists \(C > 0\), independent of \(s\), such that
\[
0 \geq \frac{\cos \alpha - \cos(\alpha/2)}{\sqrt{s + 4 \sin^2(\alpha/2)}} \geq -\frac{C \alpha^2}{\sqrt{1 + s}} \quad \text{for all} \quad \alpha \in [0, \pi].
\]
We obtain (164) by integrating this inequality and combining with (165). \(\square\)

Next we prove a similar result for \(K_{\text{axs}}\).

**Lemma 19.**
\[
r'K_{\text{axs}}(r, z, r', z') = \frac{1}{2\pi} F_1(s) \frac{(r, z) - (r', z')} {r\sqrt{rr'}} + \frac{1}{2\pi} F_2(s) \sqrt{r'}(0),
\]
where
\[
s = s(r, z, r', z') = \sqrt{(r - r')^2 + (z - z')^2},
\]
\[
F_1(s) = \frac{2}{s^2\sqrt{s^2 + 4}} - \frac{3}{8} \log\left(\frac{2 + \sqrt{4 + s^2}}{s}\right) + \frac{3}{4} \frac{1}{\sqrt{4 + s^2}} + O(\frac{1}{1 + s^3})
\]
and
\[
F_2(s) = \frac{1}{2} \log\left(\frac{2 + \sqrt{4 + s^2}}{s}\right) - \frac{1}{\sqrt{4 + s^2}} + O(\frac{1}{1 + s^3}).
\]

**Proof.** We rewrite the definition (157) as
\[
r'K_{\text{axs}}(r, z, r', z') = \frac{1}{2\pi} F_1(s) \sqrt{rr'} + \frac{1}{2\pi} F_2(s) \sqrt{r'}(0),
\]
where
\[
F_1(s) = \int_0^\pi \frac{\cos \alpha}{[s^2 + 2(1 - \cos \alpha)]^{3/2}} \, d\alpha,
\]
\[
F_2(s) = \int_0^\pi \frac{1 - \cos \alpha}{[s^2 + 2(1 - \cos \alpha)]^{3/2}} \, d\alpha.
\]
So we only need to estimate these integrals. We compute
\[
F_1(s) = \int_0^\pi \frac{\cos \alpha}{[s^2 + 4 \sin^2(\alpha/2)]^{3/2}} \, d\alpha
\]
\[
= \int_0^\pi \frac{\cos(\alpha/2)}{[s^2 + 4 \sin^2(\alpha/2)]^{3/2}} \, d\alpha + \int_0^\pi \frac{(\cos \alpha - \cos(\alpha/2))}{[s^2 + 4 \sin^2(\alpha/2)]^{3/2}} \, d\alpha.
\]
One can check that
\[
\int_0^\pi \frac{\cos(\alpha/2)d\alpha}{[s^2 + 4 \sin^2(\alpha/2)]^{3/2}} = \int_0^2 \frac{dt}{s^2\sqrt{t^2 + 1}} = \frac{t}{s^2\sqrt{s^2 + 1}} \bigg|_0^2 = \frac{2}{s^2\sqrt{s^2 + 4}}.
\]
And by examining Taylor series, one see that
\[
\cos \alpha - \cos(\alpha/2) = -\frac{3}{2} \sin^2(\alpha/2) \cos(\alpha/2) + O(\alpha^4).
\]
It follows that
\[
\int_0^\pi \frac{(\cos \alpha - \cos(\alpha/2)) \, d\alpha}{(s^2 + 4 \sin^2(\alpha/2))^{3/2}} = \frac{3}{2} \int_0^\pi \frac{\sin^2(\alpha/2) \cos(\alpha/2) \, d\alpha}{(s^2 + 4 \sin^2(\alpha/2))^{3/2}} + \int_0^\pi \frac{O(\alpha^4) \, d\alpha}{(s^2 + 4 \sin^2(\alpha/2))^{3/2}}
\]
\[
= -\frac{3}{8} \int_0^\pi \frac{t^2 \, dt}{(s^2 + t^2)^{3/2}} + O\left(\frac{1}{1 + s^3}\right).
\]
And
\[
-\frac{3}{8} \int_0^2 \frac{t^2 \, dt}{(s^2 + t^2)^{3/2}} = -\frac{3}{8} \int_0^2 \left(\frac{1}{(s^2 + t^2)^{1/2}} - \frac{s^2 \, dt}{(s^2 + t^2)^{3/2}}\right) \, dt
\]
\[
= -\frac{3}{8} \log\left(\frac{2 + \sqrt{1 + s^2}}{s}\right) + \frac{3}{4} \frac{1}{\sqrt{1 + s^2}}.
\]
Similarly,
\[
F_2(s) = \int_0^\pi \frac{1 - \cos \alpha}{|s^2 + 2(1 - \cos \alpha)|^{3/2}} \, d\alpha
\]
\[
= \int_0^\pi \frac{2 \sin^2(\alpha/2) \cos(\alpha/2) + O(\alpha^4)}{|s^2 + 4 \sin^2(\alpha/2)|^{3/2}} \, d\alpha
\]
\[
= \frac{1}{2} \log\left(\frac{2 + \sqrt{1 + s^2}}{s}\right) - \frac{1}{\sqrt{1 + s^2}} + O\left(\frac{1}{1 + s^3}\right).
\]

\[\square\]

2. steady vortex rings

By a steady vortex ring we mean a solution of (142) of the form
\[U(r, z, t) = u(r, z - vt) + ve_z\]
for some \(v \in \mathbb{R}\), and such that \(U(r, z, t) \to 0\) as \((r, z) \to \infty\), for every \(t\), and for which the vorticity is concentrated near a ring around the \(z\) axis.

Note that for \(U\) of the above form,
\[(\partial_t U + U \cdot \nabla U)(r, z, t) = (u \cdot \nabla u)(r, z - vt).
\]
Thus, written in terms of \(u\), the equations become

**VR1** (166)
\[
\begin{align*}
&u \cdot \nabla u + \nabla p = 0, \\
&\nabla \cdot (ru) = 0
\end{align*}
\]
in \(\mathbb{H}\)

with

**VR2** (167)
\[
\begin{align*}
u \cdot v &= 0 \text{ on } \partial \mathbb{H}, \\
u(r, z) &\to -ve_z \text{ as } (r, z) \to \infty.
\end{align*}
\]

Defining as usual the vorticity \(\omega(r, z) = \nabla \times u(r, z) = \nabla \times U(r, z, 0)\), one checks via easy calculations like those above that the vorticity equation (146) reduces to
\[
u \cdot \nabla \left(\frac{\omega}{r}\right) = 0, \quad u = U(r, z, 0) - ve_z = X\omega - ve_z.
\]

Note that \(\mathcal{E}\) just expressed the kinetic energy \(\int r|u|^2 \, dr \, dz\) as a function of \(\zeta = \omega/r\).

As we did with corotating vortices on \(\mathbb{R}^2\), we will find steady vortex rings by solving a suitable variational problem involving the vorticity. Due to the form of the vorticity equation, it turns out to be better to formula this problem in terms of
\[
\zeta(r, z) = \frac{\omega(r, z)}{r}.
\]
We will use the notation

\[ G_\zeta(r, z) := \int_H G_{axs}(r, z, r', z')(t' \zeta(r', z')) rr' dr' dz', \]

\[ \mathcal{E}(\zeta) := \frac{1}{2} \int_H \zeta(r, z) G_\zeta(r, z) r dr dz, \]

and

\[ \Sigma_\varepsilon := \left\{ \zeta : \int_H \zeta(r, z) r dr dz = \int_H r^2 \zeta(r, z) r dr dz = 1, \quad 0 \leq \zeta \leq \frac{1}{\pi \varepsilon^2} \text{ a.e.} \right\}. \]

There are many existence results for steady vortex rings. One is stated below.

**Theorem 8.** If \( \varepsilon \) is sufficiently small, then there exists \( \zeta_\varepsilon \) maximizing \( \mathcal{E} \) in \( \Sigma_\varepsilon \), and the following hold:

- \( \zeta_\varepsilon \) has compact support in \( \mathbb{H} \)
- \( \zeta_\varepsilon(r, z) = \zeta_\varepsilon(r, -z) \)
- \( \zeta_\varepsilon \) has the form \( \zeta_\varepsilon = \frac{1}{\pi \varepsilon^2} 1_{A_\varepsilon} \) for some \( A_\varepsilon \subset \mathbb{H} \).
- There exists \( v, \gamma \in \mathbb{R} \) such that if we define \( u_\varepsilon := \frac{1}{r} \nabla_\perp (G_\zeta - \frac{v}{2} r^2 - \gamma) \)

then \( u_\varepsilon \) solves \((166), (167)\), and

\[ A_\varepsilon = \{(r, z) \in \mathbb{H} : (G_\zeta(r, z) - \frac{v}{2} r^2 - \gamma > 0) \}\]

We will omit the proof. But here are some remarks:

- The issues that arise are quite similar to those in the proof of Theorem 2 and a proof can be given along rather similar lines.

In fact, the maximizer \( \zeta_\varepsilon \) from Theorem 8 is a maximizer in its rearrangement class, provided we consider rearrangements that preserve integrals with respect to \( r dr dz \), the natural measure in this setting.

That is, given \( \zeta_1, \zeta_2 : \mathbb{H} \to [0, \infty) \) such that \( \int_H \zeta_j(r, z) r dr dz < \infty \) for \( j = 1, 2 \), we say that \( \zeta_2 \) is a rearrangement of \( \zeta_1 \) if

\[ \int_{\{(r, z) \in \mathbb{H} : \zeta_1(r, z) \geq \lambda \}} r dr dz = \int_{\{(r, z) \in \mathbb{H} : \zeta_2(r, z) \geq \lambda \}} r dr dz \]

for every \( \lambda \geq 0 \). The rearrangement class of \( \zeta \), denoted \( R(\zeta) \) is the set of all of its rearrangements.

- One can prove that if \( \zeta_\varepsilon \) maximizes \( \mathcal{E} \) in

\[ \tilde{\Lambda}_\varepsilon(\eta) \in R(\zeta) : \int_H r^2 \eta(r, z) r dr dz = a \]

then \( \tilde{\zeta}_\varepsilon \) generates a steady vortex ring solution of \((166), (167)\). In particular this applies to \( \zeta_\varepsilon \) from Theorem 8.

- An alternate approach to proving existence the existence of steady vortex rings is directly study the problem of maximizing \( \mathcal{E} \) in \( \tilde{A}_\varepsilon \).
3. time-dependent vortex rings

By a *time-dependent vortex ring* we mean a solution of the Euler equations for axisymmetric flow without swirl

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= 0 \\
\frac{1}{r} \nabla \cdot (ru) &= 0
\end{align*}
\]

on \( \mathbb{H} \)

such that at every \( t \), the vorticity is concentrated near a ring around the \( z \) axis (but in general not evolving via a rigid motion). Thus, we study the equations for initial data \( u^0 \) such that \( \omega^0 = \nabla^\perp \cdot u^0 \approx \gamma \delta_{(r_0, z_0)} \) for some nonzero \( \gamma \in \mathbb{R} \) and \( (r_0, z_0) \in \mathbb{H} \).

For concreteness, we will suppose that \( \gamma = r_0 = 1 \). This does not involve any sacrifice of generality, since the general situation may be reduced to this one by a change of variables. Having made this assumption, it is reasonable to consider concentrations of vorticity \( \omega \) such that

\[
\int_{\mathbb{H}} \omega(r, z) \, dr \, dz = \int_{\mathbb{H}} r^2 \omega(r, z) \, dr \, dz = 1.
\]

In this section we will write the kinetic energy \( E \) as a function of \( \omega \) as in rather than of \( \zeta = \omega/r \). Thus we define

\[
\mathcal{E}(\omega) := \pi \int_{\mathbb{H} \times \mathbb{H}} G_{\text{axs}}(r, z, r', z') \omega(r, z) \omega(r', z') \, r \, r' \, dr \, dz \, dr' \, dz'
\]

where \( G_{\text{axs}} \) is introduced in (155).

3.1. vorticity concentration. The following proposition contains a “vorticity concentration” result that will be crucial for what follows.

**Proposition 4.** Assume that \( \omega : \mathbb{H} \to [0, \infty) \) satisfies (168) and

\[
0 \leq \frac{\omega(x)}{r} \leq \frac{1}{\varepsilon^2}
\]

for some \( \kappa \) independent of \( \varepsilon \). Then there exists \( C = C(\kappa) \) such that there exists \( x_\ast = (r_\ast, z_\ast) \in \mathbb{H} \) such that

\[
(1 - r_\ast)^2 \leq C \left| \frac{\omega}{\log \varepsilon} \right| \quad \text{and} \quad \int_{B_R(x_\ast)} \omega \, dr \, dz \geq 1 - \frac{C}{\log R}
\]

for every \( R \in [1, 1/\varepsilon] \).

It is an exercise to (see Section 3.1.1) to prove that for suitable choices of \( \kappa \), there are functions \( \omega \) that satisfy the hypotheses of the lemma for every sufficiently small \( \varepsilon > 0 \).

We start with a couple of lemmas. We will use the notation \( x = (r, z) \) and \( x' = (r', z') \). First we relate \( \mathcal{E}(\omega) \) to an energy-like quantity whose structure is a little more transparent. Here we require only upper bounds on the integrals in (168), rather than equality/
Lemma 20. If \( \omega : \mathbb{H} \to [0, \infty) \) satisfies
\[
\int_{\mathbb{H}} \omega(r, z) \, dr \, dz \leq 1, \quad \int_{\mathbb{H}} r^2 \omega(r, z) \, dr \, dz \leq 1.
\]
then
\[
\mathcal{E}[\omega] \leq \mathcal{E}_1[\omega] + C
\]
for
\[
\mathcal{E}_1[\omega] := \frac{1}{2} \int_{x \in \mathbb{H}} \int_{(x' \in \mathbb{H} : |x-x'| \leq 1)} \sqrt{rr'} \omega(x) \omega(x') \log_+ \frac{1}{|x-x'|} \, dx \, dx'
\]
where \( \log_+ s = \max(\log s, 0) \).

Proof. We will use the abbreviations
\[
b = |x-x'|, \quad s = \frac{1}{rr'}|x-x'|^2 = \frac{b^2}{rr'}.
\]
From (162) and (164) we find that
\[
rr'G_{\alpha\beta}(r, z, r', z') = \frac{\sqrt{rr'}}{2\pi} \left( \log 2 + \frac{\sqrt{s+4}}{\sqrt{s}} + O(1) \right).
\]
Note that
\[
\log \frac{2 + \sqrt{s+4}}{\sqrt{s}} \leq \log \frac{2\sqrt{s+4}}{\sqrt{s}} = \log 2 + \frac{1}{2} \log(1 + \frac{4rr'}{b^2}).
\]
If \( 0 < b \leq 1 \) then
\[
\log(1 + \frac{4rr'}{b^2}) \leq \log(\frac{1 + 4rr'}{b^2}) \leq \log(1 + 4rr') - 2 \log b \leq 4rr' - 2 \log b,
\]
so we conclude that
\[
\mathcal{E}[\omega] \leq \frac{1}{2} \int_{\mathbb{H} \times \mathbb{H}} \sqrt{rr'} \omega(x) \omega(x') (\log 2 + 2rr' + 1_{0 < b \leq 1 |\log b|}) \, dx \, dx'.
\]
Using the inequalities \( (rr')^{1/2} \leq 1 + r^2 r'^2 \) and \( (rr')^{3/2} \leq 1 + r^2 r'^2 \), as well as (168), we find that
\[
\mathcal{E}[\omega] \leq \mathcal{E}_1[\omega] + C(M_0^2 + M_2^2) \leq \mathcal{E}_1[\omega] + C
\]
\( \square \)

Lemma 21. If \( \omega : \mathbb{H} \to [0, \infty) \) satisfies (173) and (170), then for any \( R \geq 1 \),
\[
\mathcal{E}_1[\omega] - \frac{1}{2} |\log \epsilon| \leq \frac{1}{2} \int_{\mathbb{H} \times \mathbb{H}} \sqrt{rr'} (\log \epsilon + \log_+ \frac{1}{|x-x'|}) \omega(x) \, dx \, dx'
\]
and thus \( \mathcal{E}_1[\omega] \leq \frac{1}{2} |\log \epsilon| + C. \)
Thus (173) implies that
\[
\int_{\mathbb{H} \times \mathbb{H}} \sqrt{rr'} \omega(x)\omega(x') \, dx \, dx' \leq \int_{\mathbb{H} \times \mathbb{H}} \frac{1}{2} + \frac{1}{4} (r^2 + r'^2) \omega(x)\omega(x') \, dx \, dx' \tag{174}
\]
\[
\leq 1.
\]

Thus the definition of \(\mathcal{E}_1\) implies that
\[
\mathcal{E}_1[\omega] - \frac{|\log \epsilon|}{2} \leq \int_{\mathbb{H} \times \mathbb{H}} \sqrt{rr'} \omega(x)\omega(x') \, dx \, dx' - \frac{|\log \epsilon|}{2} \int_{\mathbb{H} \times \mathbb{H}} \sqrt{rr'} \omega(x)\omega(x') \, dx \, dx'.
\]
\[
= \frac{1}{2} \int_{\mathbb{H} \times \mathbb{H}} \sqrt{rr'} \log \frac{1}{|x-x'|} \omega(x') \, dx' \, dx.
\]

This proves the first inequality in (174), and the second inequality is clear, since the last integrand is negative whenever \(|x-x'| \geq \epsilon\). For the same reason, for every \(R \geq 1\)
\[
\int_{\{|x-x'| \leq R\epsilon\}} \sqrt{rr'} \log \frac{1}{|x-x'|} \omega(x') \, dx' \leq \int_{\{|x-x'| \leq R\epsilon\}} \sqrt{rr'} \log \frac{\epsilon}{|x-x'|} \omega(x') \, dx'.
\]

If \(|x-x'| \leq \epsilon\) then \(r' \leq r + \epsilon\) and \(\sqrt{rr'} \leq r + \epsilon\), so we use (170) and an explicit computation of an integral to estimate
\[
\int_{\{|x-x'| \leq R\epsilon\}} \sqrt{rr'} \log \frac{\epsilon}{|x-x'|} \omega(x') \, dx' \leq \frac{\pi}{2} (r + \epsilon)^2 \leq \pi (r^2 + \epsilon^2)
\]

Thus (173) implies that
\[
\frac{1}{2} \int_{\{|x-x'| \leq R\epsilon\}} \sqrt{rr'} \log \frac{1}{|x-x'|} \omega(x') \, dx' \, dx \leq \frac{\pi}{2} (1 + \epsilon^2).
\]

In view of the previous lemma, we see that the key hypothesis (171) of the Proposition states that \(\mathcal{E}[\omega]\), and hence \(\mathcal{E}_1[\omega]\), are close to maximal, given the constraints (168) and (170). We put this information to use in completing the proof of the Proposition.

**Proof of Proposition 4. Step 1.** It follows from (171) and Lemma 20 that
\[
\mathcal{E}_1[\omega] \geq \frac{1}{2} |\log \epsilon| - C.
\]

Since
\[
\int_{\mathbb{H} \times \mathbb{H}} \frac{1}{2} + \frac{1}{4} (r^2 + r'^2) \omega(x)\omega(x') \, dx \, dx' \leq 1,
\]
we can rewrite (175) as
\[
\frac{|\log \epsilon|}{2} \int_{\mathbb{H} \times \mathbb{H}} \frac{1}{2} + \frac{1}{4} (r^2 + r'^2) \omega(x)\omega(x') \, dx \, dx' \leq \mathcal{E}_1[\omega] + C.
\]
We subtract $\frac{1}{2} \int_{\mathbb{H} \times \mathbb{H}} \sqrt{rr'} \omega(x) \omega(x') \, dx \, dx'$ from both sides to obtain

\begin{equation}
\log \rho f \quad (176) \quad 0 \leq \frac{1}{2} \int_{\mathbb{H} \times \mathbb{H}} f(r, r') \omega(x) \omega(x') \, dx \, dx'
\end{equation}

\begin{equation}
\leq \frac{1}{2} \int_{\mathbb{H} \times \mathbb{H}} \sqrt{rr'} \left( \log \epsilon + \log_{+} \frac{1}{|x - x'|} \right) \omega(x) \omega(x') \, dx \, dx' + C
\end{equation}

for

\begin{equation}
f(r, r') := \frac{1}{2} + \frac{1}{4} (r^2 + r'^2) - \sqrt{rr'} = \frac{1}{2} (1 - \sqrt{rr'})^2 + \frac{1}{4} (r - r')^2.
\end{equation}

We remark for later use that

\begin{equation}
f \cdot \mathrm{prop} \quad (177) \quad 1 \leq 2 |f(r, r') + \sqrt{rr'}| \quad \text{in } \mathbb{H} \times \mathbb{H}
\end{equation}

and it is straightforward to check that there exists a constant $C_\ast > 0$ such that

\begin{equation}
f \cdot \mathrm{prop} \quad (178) \quad (1 - r)^2 \leq C_\ast f(r, r') \quad \text{for } (x, x') \in \mathbb{H} \times \mathbb{H} \text{ such that } f(r, r') \leq 1.
\end{equation}

Indeed, this follows from the facts that

- $\{(r, r') \in [0, \infty) \times [0, \infty) : f(r, r') \leq 1\}$ is compact.
- $f > 0$ except at $(1, 1)$.
- $\partial_2^2 f(1, 1) > 0$.

All of these are easy to verify.

Step 2. It follows from (176) and Lemma [21] that

\begin{equation}
\log \rho f \quad (179) \quad \int_{\mathbb{H} \times \mathbb{H}} f(r, r') \omega(x') \omega(x) \, dx \, dx' \leq \frac{C}{|\log \epsilon|} \leq \frac{C}{\log R}
\end{equation}

Next, we have seen in Lemma [21] that for any $R \geq 1,$

\begin{equation}
\frac{1}{2} \int_{\{|x - x'| \in R\}} \sqrt{rr'} \left( \log \epsilon + \log_{+} \frac{1}{|x - x'|} \right) \omega(x) \omega(x') \, dx \, dx' \leq C.
\end{equation}

We subtract this from the lower bound in (176) for the integral over $\mathbb{H} \times \mathbb{H}$ of the same quantity to find that

\begin{equation}
\geq \frac{1}{2} \int_{\{|x - x'| > R\}} \sqrt{rr'} \left( \log \epsilon + \log_{+} \frac{1}{|x - x'|} \right) \omega(x) \omega(x') \, dx \, dx' \geq -C.
\end{equation}

Despite its appearance, this should be thought of as an upper bound, since both sides are negative. Indeed, since

\begin{equation}
(\log \epsilon + \log_{+} \frac{1}{|x - x'|}) \leq -\log R \quad \text{if } |x - x'| \geq R \epsilon \text{ and } R \geq 1,
\end{equation}

it follows that

\begin{equation}
frice \quad (180) \quad -\frac{\log R}{2} \int_{\{|x - x'| > R\}} \sqrt{rr'} \omega(x) \omega(x') \, dx \, dx' \geq -C,
\end{equation}

and thus

\begin{equation}
\int_{\{|x - x'| > R\}} \sqrt{rr'} \omega(x) \omega(x') \, dx \, dx' \leq \frac{C}{|\log R|}.
\end{equation}

Step 3. By taking a suitable linear combination of (179) and (180) and using (177), we deduce that

\begin{equation}
\int_{\mathbb{H} \times \mathbb{H}} \left[ 1_{|x - x'| \geq R \epsilon} + C_\ast f(r, r') \right] \omega(x') \omega(x) \, dx \, dx' \leq \frac{C}{|\log R|}
\end{equation}
for \( C \), as in (178). Since \( \int_\Sigma \omega(x)dx = 1 \), it follows that there must exist some \( x = (r, z) \in \Sigma \) such that

\[
\int_{[x-x']|\geq R_\varepsilon} |C_x f(r, r')| \omega(x') \, dx' \leq \frac{C}{\log R}.
\]

For this \( x \), it follows that

\[
\int_{[x-x']|< R_\varepsilon} \omega(x') \, dx' = 1 - \int_{[x-x']|\geq R_\varepsilon} \omega(x') \, dx' \geq 1 - \frac{C}{\log R}.
\]

Moreover, (181) also implies that there exists \( x' \) such that \( C_x f(r, r') \leq \frac{C}{\log R} \), which according to (178) implies that

\[
(1 - r)^2 \leq \frac{C}{\log R}.
\]

**Step 4.** So far we have shown that for any \( R \in [1, \frac{1}{2}] \), there exists some \( x = (r, z) \), possibly depending on \( R \), such that (182), (183) hold. We now want to eliminate the dependence on \( R \), at the expense of adjusting the constants \( C \) if necessary. This just involves messing around with details to get a slightly nicer-looking conclusion. I would say it is conceptually clear that this can be done, and I recommend looking at the details only if you really care about this issue.

Since we will need to keep track of constants, let \( C_1 = C_1(\kappa) \) denote the constant in (182). We first fix \( R_0 \) such that \( \frac{C_1}{\log R_0} = \frac{1}{2} \), i.e. \( R_0 = \exp[2C_1] \) and we let \( x_* \) denote a point satisfying (182), (183) with \( R = R_0 \).

Now consider any \( R \) such that \( R_0 < R \leq \frac{1}{2} \), and let \( x \) be a point satisfying (182), (183). Then

\[
\int_{B_{R_0}(x_*)} \omega(x') \, dx \geq \frac{1}{2} \quad \text{and} \quad \int_{B_{R}(x)} \omega(x') \, dx' > \frac{1}{2}.
\]

It follows that \( B_{R_0}(x_*) \cap B_{R}(x) \neq \emptyset \) and hence that \( B_{R}(x) \subseteq B_{3R}(x_*) \). Thus

\[
\int_{B_{3R}(x_*)} \omega(x') \, dx' \geq \int_{B_{R}(x)} \omega(x') \, dx' \geq 1 - \frac{C_1}{\log R} \geq 1 - \frac{2C_1}{\log 3R}
\]

as long as \( R \geq 9 \), which implies that \( \log 3R \leq 2 \log R \). Thus for this choice of \( x_* \), conclusion (182) holds for all \( R = 3R \geq 3R_0 = 3 \exp[2C_1] \). On the other hand, it is certainly true that

\[
\int_{B_{R}(x_*)} \omega(x') \, dx' \geq 1 - \frac{\log 3R_0}{\log R} = 1 - \frac{\log 3 + 2C_1}{\log R} \quad \text{whenever} \quad R < R_0.
\]

It follows that \( \int_{B_{R}(x_*)} \omega(x') \, dx' \geq 1 - \frac{C_2}{\log R} \) for all \( R \geq 1 \), as long as \( C_1 \geq 2C_1 + 2 \).

Finally, let \( x = (r, z) \) be a point satisfying (182) with \( R = \sqrt{\varepsilon} \). We have already seen that \( B_{R}(x) \cap B_{R_0}(x_*) \neq \emptyset \), so \( |x-x_*| \leq (\sqrt{\varepsilon}+R_0 \varepsilon) \). In addition, we know from (183) that \( |r-1| \leq \frac{C}{\log(\varepsilon+1)} \leq \frac{C}{\log 3 \varepsilon} \), and it follows that \( |r-1| \leq \frac{C}{\log 3 \varepsilon} \) (where we have resumed our practice of adjusting the meaning of “\( C \)” from one occurrence to the next.)
3.1.1. Some exercises. The first few exercises prove the existence of functions satisfying the hypotheses of Proposition 4.

1. Prove that for every $\varepsilon < 1$ there exists $R \in (1 - \varepsilon, 1 + \varepsilon)$ such that

$$\omega_\varepsilon := \frac{1}{\pi \varepsilon^2} 1_{B_\varepsilon((R,0))} \text{ satisfies (168).}$$

Here as usual we use the notation

$$1_{B_\varepsilon((R,0))}(r,z) = \begin{cases} 
1 & \text{if } |r - R|^2 + z^2 < \varepsilon^2 \\
0 & \text{if not.}
\end{cases}$$

2. Let $\omega_\varepsilon$ denote the function from the previous exercise. Since $R \geq 1 - \varepsilon$, it is clear that $r \geq 1 - 2\varepsilon$ for $(r,z) \in \text{supp}(\omega_\varepsilon)$. Using this fact, find a lower bound for $E_1[\omega_\varepsilon]$. I believe you should be able to prove that

$$E_1[\omega_\varepsilon] \geq (1 - 2\varepsilon)^{\frac{1}{2}} |\log \varepsilon|.$$

Hint: The main computation you will need to do is similar to one we have done earlier. So one way to proceed is to find that earlier lemma and cite it. The proof of that lemma omitted the details, because I did not feel like typing them, but if you wish, you can also fill in those details.

3. For the same $\omega_\varepsilon$, show that there exists a constant $\kappa_1$ such that

$$E[\omega_\varepsilon] \geq E_1[\omega_\varepsilon] - \kappa_1.$$

Hint: Look at the proof of Lemma 20.

The next few exercises demonstrate how one can use a change of variables to establish mild generalizations of the results proved above. in which we assume

$$\int_\mathbb{R} \omega(x) \, dx \leq \gamma, \quad \int_\mathbb{R} r^2 \omega(x) \, dx \leq \gamma R^2$$

for positive numbers $\gamma, R$, rather than the special case (173). These assumptions are consistent with $\omega \approx \gamma \delta_{(R,z_0)}$ for some $z_0 \in \mathbb{R}$. (Similar arguments would also apply with equality rather than $\leq$ in (173).)

4. Assume that $\omega : \mathbb{H} \to [0, \infty)$ satisfies (184).

(a) Prove that

$$\tilde{\omega}(x) := \frac{R^2}{\gamma} \omega(Rx) \text{ satisfies (168).}$$

(b) Prove that $E[\tilde{\omega}] = \frac{1}{R \gamma^2} E[\omega]$.

(c) Prove that

$$\sup_{\mathbb{H}} \frac{\omega(x)}{r} \leq \frac{1}{\varepsilon^2} \iff \sup_{\mathbb{H}} \frac{\tilde{\omega}(x)}{r} \leq \frac{R^3}{\gamma \varepsilon^2}$$

where as usual $x = (r,z)$.

(d) Assume that $\omega : \mathbb{H} \to [0, \infty)$ satisfies (184) and that $0 \leq \frac{\omega(x)}{r} \leq \frac{1}{\varepsilon^2}$. By rescaling some of the results proved above, show that

$$E[\omega] \leq \frac{1}{4} R \gamma^2 \log\left(\frac{R^3}{\gamma \varepsilon^2}\right)$$
Our main result below shows that the vortex ring structure (circulation, radius, cross-section) is preserved by the Euler equations (this is almost immediate from Lemma 4) and that the ring moves with velocity $\frac{1}{\pi |\log \epsilon|}$ in the positive $z$ direction.

Simple rescaling arguments then show that a ring with radius $\tau_0$, circulation $\gamma \in \mathbb{R}$ and cross-section $\approx \epsilon$ maintains its shape and moves with velocity $\frac{\gamma}{4\pi \tau_0} |\log \epsilon|$ in the $z$ direction.

Since the velocity is $O(|\log \epsilon|)$, it is useful to introduce a new time variable $\tau = |\log \epsilon| t$, so that $|\log \epsilon| \partial_\tau = \partial_t$. In terms of the $\tau$ variable, the equations may be written in the form

\begin{equation}
\left|\log \epsilon\right| \partial_\tau u_\epsilon + \frac{1}{r} \nabla \cdot \left( ru_\epsilon \otimes u_\epsilon \right) + \nabla p_\epsilon = 0, \quad \nabla \cdot (ru_\epsilon) = 0.
\end{equation}

Our main result is this:

**Theorem 9.** Assume that $u_\epsilon : \mathbb{H} \times [0, \infty) \rightarrow [0, \infty)$ solves the time-rescaled axisymmetric Euler without swirl (187) for initial data $u_0^\epsilon$ such that $\omega_0^\epsilon = \nabla \perp \cdot u_0^\epsilon$ satisfies (180), for some $\kappa$ independent of $\epsilon$.

Then there exists $C = C(\kappa)$ such that for every $\epsilon \in (0, 1]$ and $\tau > 0$ there exists $x_{\epsilon, \tau}(\tau) \in \mathbb{H}$ such that

\begin{equation}
\left(1 - r_{\epsilon, \tau}(\tau)^2\right)^2 \leq \frac{C}{|\log \epsilon|}
\end{equation}

\begin{equation}
\int_{\{x \in \mathbb{H} : \left| x_{\epsilon, \tau}(\tau) - x \right| < R\}} \omega_\epsilon(x, t) \, dr \, dz \geq 1 - \frac{C}{\log R} \quad \text{for every } R \in [1, 1/\epsilon]
\end{equation}

where $C = C(\kappa)$. Moreover, for any $\tau > 0$,

\begin{equation}
z_{\epsilon, \tau}(\tau) - z_{\epsilon, \tau}(0) \to -\frac{1}{4\pi} \tau \quad \text{as } \epsilon \to 0.
\end{equation}
Lemma 22. For $\phi \in C^1_c(\mathbb{R})$,

$$\int_{\mathbb{R}} \phi(x) \omega_\varepsilon(x, \tau) \, dx = \phi(x_{\varepsilon, *} (\tau)) + O(|\log \varepsilon|^{-1})$$

where the implicit constants in the $O(|\log \varepsilon|^{-1})$ term depend on $\|\phi\|_{C^1}$.

Here and below, for integer $k \geq 0$ we use the notation

$$\|\phi\|_{C^k} = \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}} |D^\alpha \phi(x)|$$

where the right-hand side denotes the sum over all partial derivatives of order at most $k$ (including of order 0, that is $\phi$ itself).

\footnote{A minor subtlety is that $x_{\varepsilon, *}$ is not uniquely determined by conditions (189), (188). But it is easy to see that these conditions determine the position of $x_{\varepsilon, *}$ up to possible deviations of size at most $R_0 \varepsilon$, for some $R_0$ depending on $\kappa$ but independent of $\varepsilon$. Anyway, we may assume, and we do, that we have fixed some measurable function $\tau \mapsto x_{\varepsilon, *} (\tau)$ satisfying (189), (188) for every $\tau$.}
3. TIME-DEPENDENT VORTEX RINGS

Proof. For any \( \tau \),
\[
\int_{\mathbb{H}} \phi(x) \omega_{\varepsilon}(x, \tau) \, dx - \phi(x_{\varepsilon, \tau}) = \int_{\mathbb{H}} \phi(x) - \phi(x_{\varepsilon, \tau}) \omega_{\varepsilon}(x, \tau) \, dx
\]
\[
= \int_{B_{\sqrt{\varepsilon}}(x_{\varepsilon, \tau})} (\cdots) \, dx + \int_{B_{\sqrt{\varepsilon}}(x_{\varepsilon, \tau})^c} (\cdots) \, dx.
\]
Clearly
\[
|\phi(x) - \phi(x_{\varepsilon, \tau})| \leq \|\phi\|_{C^1} \sqrt{\varepsilon}
\]
for \( x \in B_{\sqrt{\varepsilon}}(x_{\varepsilon, \tau}) \) so we deduce from this and (189) that
\[
\int_{B_{\sqrt{\varepsilon}}(x_{\varepsilon, \tau})} (\cdots) \, dx + \int_{B_{\sqrt{\varepsilon}}(x_{\varepsilon, \tau})^c} (\cdots) \, dx \leq C \|\phi\|_{C^1} \sqrt{\varepsilon} + C \|\phi\|_{C^0} |\log \varepsilon|^{-1}.
\]
\[\Box\]

Next we establish a very crude estimate of the terms in (193).

Lemma 23. Assume that \( \phi \in C^2_\varepsilon(\mathbb{H}) \) and that \( \text{supp}(\phi) \subset \{(r, z) : r \geq r_0\} \) for some \( r_0 > 0 \). Then
\[
\left| \frac{d}{d\tau} \int_{\mathbb{H}} \phi(x) \omega_{\varepsilon}(x, \tau) \, dx \right| \leq C(r_0) \|\phi\|_{C^2}.
\]

Proof. The right-hand side of (193) is bounded by
\[
\int_{\mathbb{H}} \frac{|D^2\phi|}{\tau} \left| r|u_{\varepsilon}|^2 \frac{1}{|\log \varepsilon|} + (|\nabla^\perp \phi||\nabla \frac{1}{r}|) \frac{|r|u_{\varepsilon}|^2}{|\log \varepsilon|} \right| \, dr \, dz.
\]
Then we use (191) to estimate
\[
\left| \frac{d}{d\tau} \int_{\mathbb{H}} \phi \omega_{\varepsilon} \, dx \right| \leq C(r_0) \|\phi\|_{C^2} \int_{\mathbb{H}} \frac{r|u_{\varepsilon}|^2}{|\log \varepsilon|} \, dr \, dz \leq C(r_0) \|\phi\|_{C^2}.
\]
\[\Box\]

We now define a couple of particular test functions of the form
\[
\phi_{j}(r, z) = f(r)g_{j}(z), \quad j = 1, 2.
\]
For both \( j = 1, 2 \) we take
\[
\text{supp} f \subset \left[ \frac{1}{3}, 3 \right], \quad f(r) = 1 \text{ for } \frac{2}{3} \leq r \leq \frac{3}{2}.
\]
We specify that
\[
g_{1}(z) = z - z_{\varepsilon, \star}(0) \quad \text{if} \quad |z - z_{\varepsilon, \star}(0)| \leq 2, \quad \text{supp} \, g_1 \text{ is compact}
\]
and
\[
g_{2}(z) = \begin{cases} 1 & \text{if} \quad |z - z_{\varepsilon, \star}(0)| \leq 1 \\ 0 & \text{if} \quad |z - z_{\varepsilon, \star}(0)| \geq 2. \end{cases}
\]
We will write
\[
A = \{(r, z) : \frac{2}{3} \leq r \leq \frac{3}{2}, \quad |z - z_{\varepsilon, \star}(0)| \leq 2\}
\]
\[
B := \text{supp}(\phi_1) \setminus A.
\]
LEMMA 24. There exists $T > 0$ and $\varepsilon_0 > 0$ such that for
$$x_{\varepsilon, s}(\tau) \in A, \text{ and } \text{dist}(x_{\varepsilon, s}(\tau), B) \geq \frac{1}{10}$$
for $0 \leq \tau \leq T$ and $0 < \varepsilon \leq \varepsilon_0$.

PROOF. First, owing to (188), we may insist that $\varepsilon_0 > 0$ is small enough that
$$|x_{\varepsilon, s}(\tau) - 1| < \frac{1}{10} \quad \text{for all } \tau, \text{ whenever } 0 < \varepsilon < \varepsilon_0.$$  

Next, we will write
$$Z_{1,\varepsilon}(\tau) := \int_{\Omega} \phi_1(x) \omega_\varepsilon(x, \tau) \, dx$$
Clearly
$$|Z_{1,\varepsilon}(\tau)| \leq |Z_{1,\varepsilon}(\tau) - Z_{1,\varepsilon}(0)| + |Z_{1,\varepsilon}(0)|$$
It follows from Lemma 23 that
$$|Z_{1,\varepsilon}(\tau) - Z_{1,\varepsilon}(0)| \leq C|\tau|$$
and we deduce from Lemma 22 (196) and the definition of $\phi_1$ that
$$Z_{1,\varepsilon}(0) = \phi_1(x_{\varepsilon, s}(0)) + O(|\log \varepsilon|^{-1}) = g_1(z_{\varepsilon, s}(0)) + O(|\log \varepsilon|^{-1}) = O(|\log \varepsilon|^{-1}).$$
Thus, by fixing $T$ and $\varepsilon_0$ small enough, we deduce that
$$|Z_{1,\varepsilon}(\tau)| \leq C\tau + O(|\log \varepsilon|^{-1}) \leq \frac{3}{4} \quad \text{whenever } 0 \leq \tau \leq T \text{ and } 0 < \varepsilon \leq \varepsilon_0.$$
Again appealing to Lemma 22 and (196), we see that for such $\tau$ and $\varepsilon$
$$|g_1(z_{\varepsilon, s}(\tau))| = |\phi_1(z_{\varepsilon, s}(\tau))| \leq |Z_{\varepsilon, s}(\tau)| + O(|\log \varepsilon|^{-1}) \leq 1.$$ 
Then the definition of $g_1$ implies that
for $0 \leq \tau \leq T$, either $|z_{\varepsilon, s}(\tau) - z_{\varepsilon, s}(0)| \leq 1$ or $|z_{\varepsilon, s}(\tau) - z_{\varepsilon, s}(0)| > 2$.
Then (196), the definition of $\phi_2$, and Lemma 22 imply that for $\varepsilon$ sufficiently small,
$$|z_{\varepsilon, s}(\tau) - z_{\varepsilon, s}(0)| \leq 1 \iff \phi_2(x_{\varepsilon, s}(\tau)) = 1 \iff \int_{\Omega} \phi_2(x) \omega_\varepsilon(x, \tau) \, dx \geq \frac{3}{4}$$
and
$$|z_{\varepsilon, s}(\tau) - z_{\varepsilon, s}(0)| > 2 \iff \phi_2(x_{\varepsilon, s}(\tau)) = 0 \iff \int_{\Omega} \phi_2(x) \omega_\varepsilon(x, \tau) \, dx \leq \frac{1}{4}.$$ 
The first of these two cases holds when $\tau = 0$. Then since $\tau \mapsto \int \phi_2(x) \omega_\varepsilon(x, \tau) \, dx$ is continuous (in fact differentiable, by Lemma 23) the first case must continue to hold for all $\tau \in [0, T]$.
Thus $x_{\varepsilon, s}(\tau) \in \left[\frac{9}{10}, \frac{11}{10}\right] \times [z_{\varepsilon, s}(0) - 1, z_{\varepsilon, s}(0) + 1]$ for $0 \leq \tau \leq T$ and $0 < \varepsilon \leq \varepsilon_0$, which implies the conclusion of the Lemma.  

The Lemma implies that for $0 \leq \tau \leq T$ and $0 < \varepsilon \leq \varepsilon_0$,
$$Z_{1,\varepsilon}(\tau) = \int_{\Omega} \phi_1(x) \omega_\varepsilon(x, \tau) \, dx = z_{\varepsilon, s}(\tau) - z_{\varepsilon, s}(0) + O(|\phi|_{C_1}|\log \varepsilon|^{-1})$$
Now, armed with the knowledge provided by Lemma 24 together with (189), we will return to the basic identity (193) and obtain much more precise estimates than before. In these computations the $\tau$ variable plays no role, so we will just consider any $\omega_\varepsilon$ satisfying the relevant properties of $\omega_\varepsilon(\cdot, \tau)$ for arbitrary $\tau \in [0, T]$. 

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(197) 
$$Z_{1,\varepsilon}(\tau) = \int_{\Omega} \phi_1(x) \omega_\varepsilon(x, \tau) \, dx = z_{\varepsilon, s}(\tau) - z_{\varepsilon, s}(0) + O(|\phi|_{C_1}|\log \varepsilon|^{-1})$$
LEMMA 25. Assume that $\omega_\varepsilon$ satisfies (168), (170), (171) and that $x_{\varepsilon, *} \in \mathbb{R}^2$ is a point satisfying (188), (189). Assume in addition that $x_{\varepsilon, *}, \in \mathbb{R}^2$ and $\text{dist}(x_{\varepsilon, *}, B) \geq 1/T_0$. Then
\[
\frac{1}{\log \varepsilon} \int_{\mathbb{H}} r^2 \frac{\nabla \phi_1}{r} \cdot dx = \frac{1}{4\pi} + o(1) \quad \text{as} \quad \varepsilon \to 0.
\]

Before presenting the proof of the lemma, we use it to complete the proof of Theorem 9. It follows from identity (193) and Lemmas 24 and 25 that for every $\tau \in [0, T]$, we have
\[
\frac{d}{d\tau} Z_{1, \varepsilon}(\tau) = \frac{d}{d\tau} \int_{\mathbb{H}} \phi_1(x) \omega_\varepsilon(x, \tau) dx = -\frac{1}{4\pi} + o(1) \quad \text{as} \quad \varepsilon \to 0,
\]
where the $o(1)$ term is uniform for $\tau \in [0, T]$. Since we have shown that $Z_{1, \varepsilon}(0) \to 0$ as $\varepsilon \to 0$, it follows that $Z_{1, \varepsilon}(\tau) \to -\frac{1}{4\pi} \tau$ for $\tau \in [0, T]$. Next, it follows from Lemmas 22 and 24 that $Z_{1, \varepsilon}(\tau) = z_{\varepsilon, *}(\tau) - z_{\varepsilon, *}(0) + o(1)$ as $\varepsilon \to 0$, for $\tau \in [0, T]$. Assembling these facts, we conclude that
\[
z_{\varepsilon, *}(\tau) - z_{\varepsilon, *}(0) \to \frac{1}{4\pi} \tau \quad \text{as} \quad \varepsilon \to 0,
\]
for $\tau \in [0, T]$. This is the final conclusion of the Theorem, except that it is restricted to times $\tau \in [0, T]$. But note that $u_\varepsilon(\cdot, \tau/2)$ satisfies the hypotheses on the initial data, so by applying the theorem on the time interval $1/4 < \tau < 3/4$, we see that $z_{\varepsilon, *}(\frac{\tau}{4}) - z_{\varepsilon, *}(\frac{3}{4}) \to \frac{1}{4\pi} \tau$ as $\varepsilon \to 0$. In other words, (198) holds for $\tau \in [0, 3T/4]$. Since we can continue in this way indefinitely, we conclude that (198) holds for all $\tau > 0$. (Indeed, also for all $\tau < 0$.)

Finally, we present the proof of Lemma 25. Step 1. We will write
\[
\omega_\varepsilon^i = \omega_\varepsilon 1_{B_\varepsilon | \log \varepsilon}(x_{\varepsilon, *}) \quad \text{and} \quad \omega_\varepsilon^o = \omega_\varepsilon - \omega_\varepsilon^i
\]
and
\[
u_i = \int_{\mathbb{H}} K_{axs}(x, x') \omega_\varepsilon^o(x') r' dr' dz'; \quad \nu_i^0 = \int_{\mathbb{H}} K_{axs}(x, x') \omega_\varepsilon^i(x') r' dr' dz'.
\]
Clearly
\[
u_\varepsilon = \nu_i + \nu_i^0.
\]
Then (188) implies that
\[
\gamma^i := \int_{\mathbb{H}} \omega_\varepsilon^i(x) dx \geq 1 - \frac{C}{\log |\log \varepsilon|}, \quad \gamma^o := \int_{\mathbb{H}} \omega_\varepsilon^o(x) dx \leq \frac{C}{\log |\log \varepsilon|}.
\]
It then follows from (189) that
\[
\int_{\mathbb{H}} r^2 \omega_\varepsilon^i(x) dx \geq \gamma^i (1 - \frac{C}{\sqrt{|\log \varepsilon|}}),
\]
and thus that
\[
1 - \gamma^i (1 - \frac{C}{\sqrt{|\log \varepsilon|}}) \leq \frac{C}{\log |\log \varepsilon|}.
\]
Now it follows from \eqref{199}, \eqref{200}, and \eqref{185} that
\[\pi \int_{\mathbb{H}} r |u^o_\epsilon|^2 = \mathcal{E}[\omega^o_\epsilon] \leq \frac{C}{\log \log \epsilon^2} \left( \log \frac{\log \epsilon}{\epsilon} + C \right) \leq C \left( \frac{\log \log |\log \epsilon|}{(\log |\log \epsilon|)^2} \right) |\log \epsilon|.
\]

**Step 2.** Next,
\[
\frac{1}{|\log \epsilon|} \int_{\mathbb{H}} |\nabla (\frac{\nabla \phi_1}{r})| : u_\epsilon \otimes u_\epsilon \, dx = \frac{1}{|\log \epsilon|} \int_{\mathbb{H}} \left[ u^i_\epsilon \otimes u^i_\epsilon + u^i_\epsilon \otimes u^o_\epsilon + u^o_\epsilon \otimes u^i_\epsilon + u^o_\epsilon \otimes u^o_\epsilon \right] r \, dx
\]
We know that $|\nabla (\frac{\nabla \phi_1}{r})| \leq C \|\phi_1\|_{C^2} \leq C$ (that is, every entry of this $2 \times 2$ matrix is bounded by the right-hand side). Thus
\[
\frac{1}{|\log \epsilon|} \int_{\mathbb{H}} |\nabla (\frac{\nabla \phi_1}{r})| : (u^0_\epsilon \otimes u^0_\epsilon) r \, dx \leq \frac{C}{|\log \epsilon|} \int_{\mathbb{H}} |u^0_\epsilon|^2 \, r \, dx \to 0
\]
as $\epsilon \to 0$, by Step 1. Similarly, using Cauchy-Schwarz,
\[
\frac{1}{|\log \epsilon|} \int_{\mathbb{H}} \left( |\nabla (\frac{\nabla \phi_1}{r})| : (u^i_\epsilon \otimes u^i_\epsilon) r \, dx \right) \leq \frac{C}{|\log \epsilon|} \left( \int_{\mathbb{H}} |u^0_\epsilon|^2 \, r \, dx \right)^{1/2} \to 0
\]
as $\epsilon \to 0$. The term involving $u^0_\epsilon \otimes u^i_\epsilon$ is of course handled similarly. So
\[
\frac{1}{|\log \epsilon|} \int_{\mathbb{H}} |\nabla (\frac{\nabla \phi_1}{r})| : (u_\epsilon \otimes u_\epsilon) r \, dx = \frac{1}{|\log \epsilon|} \int_{\mathbb{H}} |\nabla (\frac{\nabla \phi_1}{r})| : (u^i_\epsilon \otimes u^i_\epsilon) r \, dx + o(1)
\]
as $\epsilon \to 0$.

**Step 3.** Next, we write
\[
U^i_\epsilon = U^i_\epsilon + \tilde{U}^i_\epsilon \quad \text{where}
\]
where
\[
U^i_\epsilon = \mathcal{U}^i_\epsilon + \mathcal{U}^i_\epsilon
\]
and then
\[
\tilde{U}^i_\epsilon(x) = \int_{\mathbb{H}} \frac{(x - x')^\perp}{|x - x'|^2} \omega^i_\epsilon(x') \, dx
\]
and then
\[
\tilde{U}^i_\epsilon(x) = \int_{\mathbb{H}} \left( r' K_{\text{axs}}(x, x') - \frac{(x - x')^\perp}{|x - x'|^2} \right) \omega^i_\epsilon(x') \, dx.
\]
It follows from computations in the proof of Lemma \ref{19} that for $x, x' \in \text{supp}(\omega^i_\epsilon) \in A$,
\[
\left| r' K_{\text{axs}}(x, x') - \frac{(x - x')^\perp}{|x - x'|^2} \right| \leq C \left( 1 + \log \frac{1}{|x - x'|} \right)
\]
Thus, taking into account the supports of $\omega^i_\epsilon$ and $\phi_1$,
\[
|\tilde{U}^i_\epsilon(x)|_{1_{\text{supp}(\phi_1)}}(x) \leq \int_{\mathbb{H}} f(x - x') \omega^i_\epsilon(x') \, dx' \quad \text{for } f(x) = C \left( 1 + \log \frac{1}{|x|} \right) 1_{|x| \leq C}.
\]
\[\tag{201}
\]
\[\text{Uepi.def}
\]
\[\text{Section 3.1.1}
\]
\[\text{5obtained by rescaling estimates in Lemmas 20 and 175 as discussed in the Exercises in Section 3.1.1}
\]
Then Young’s inequality implies that
\[
\int_{\text{supp} (\phi_1)} |\bar{U}_1^i (x)|^2 \, d\mathbf{r} \, dz \leq C \int_{\text{supp} (\phi_1)} |\bar{U}_1^i (x)|^2 \, d\mathbf{r} \\
\leq C \|f \ast \omega_2^1\|_{L^2} \leq C \|f\|_{L^2} \|\omega_1^i\|_{L^2} \leq C.
\]

With this estimate, we can complete the arguments of Step 2 to find that
\[
\frac{1}{|\log \epsilon|} \int_{\mathbb{H}} |\nabla (\frac{\nabla \cdot \phi_1}{r})| : (\bar{u}_c \otimes \bar{u}_c) \, d\mathbf{r} = \frac{1}{|\log \epsilon|} \int_{\mathbb{H}} |\nabla (\frac{\nabla \cdot \phi_1}{r})| : (\bar{U}_c^i \otimes \bar{U}_c^i) \, d\mathbf{r} + o(1)
\]
as \(\epsilon \to 0\).

**Step 4.** We now split the right-hand side into several terms, using the facts (clear from the definitions of \(\phi_1\) and the sets \(A, B\), see (194) and (195)) that
\[
\text{supp} (\nabla \cdot \phi_1) \subset B, \quad ru_c \cdot \nabla (\frac{1}{r}) = -\frac{u_c}{r}, \quad \nabla \cdot \phi_1 = \left(\begin{array}{c} 0 \\ \frac{1}{r} \end{array}\right) \text{ in } A.
\]

These facts and facts from Steps 2 and 3 allow us to write
\[
\frac{1}{|\log \epsilon|} \int_{\mathbb{H}} |\nabla (\frac{\nabla \cdot \phi_1}{r})| : (\bar{u}_c \otimes \bar{u}_c) \, d\mathbf{r} = T_1 + T_2 + o(1)
\]
as \(\epsilon \to 0\).

where (writing \(U_c^i = (U_c^i, U_c^i, z)\))
\[
T_1 := \frac{1}{|\log \epsilon|} \int_{B} |\nabla (\frac{\nabla \cdot \phi_1}{r})| : (\bar{U}_c^i \otimes \bar{U}_c^i) \, d\mathbf{r}
\]
\[
T_2 := \frac{1}{|\log \epsilon|} \int_{A} \frac{1}{r} (\bar{U}_c^i \otimes \bar{U}_c^i) \, d\mathbf{r}
\]

To estimate these terms, we first observe from the definition (201) of \(U_c^i\) and Example 2 in Section 3.1 that if \(|x - x_{\epsilon,s}| \geq 2\epsilon |\log \epsilon|\), then
\[
|U_c^i (x) - K (x)| \leq \frac{1}{|\log \epsilon|} |x - x_{\epsilon,s}|^{-2} \quad \text{for } K (x) = \gamma^i \frac{(x - x_{\epsilon,s})}{2\pi |x - x_{\epsilon,s}|^2}.
\]

*Recall that \(\gamma^i = 1 - O(|\log \log \epsilon|^{-1})\) was defied above (199).* It then follows from Lemma 24 and the definition (201) of \(U_c^i\) that \(|U_c^i (x)| \leq C\) in \(B\). Thus
\[
|T_1| \leq \frac{C(\phi)}{|\log \epsilon|} \int_{B} |U_c^i|^2 \, d\mathbf{r} \leq \frac{C}{|\log \epsilon|} \rightarrow 0 \quad \text{as } \epsilon \to 0.
\]

Finally, we write \(A = A_1 := \{x \in \mathbb{H} : 2\epsilon |\log \epsilon| \leq |x - x_{\epsilon,s}| \leq \frac{1}{|\log \epsilon|}\} \) and \(A_2 = A \setminus A_1\).

Note that Lemma 24 implies that \(A_1 \subset A\). It follows from (202) and straightforward computations that (writing \(K (x) = (K^\gamma (x), K^\delta (x))\) as usual)
\[
|U_c^i - K|_{L^2 (A_2, A_1)} \leq C
\]
and
\[
|K^\gamma|_{L^2 (A_1)} = |K^\delta|_{L^2 (A_1)} = \frac{\gamma^i}{2} |K^\gamma|_{L^2 (A_1)} = \frac{1}{4\pi} \int_{2\epsilon |\log \epsilon|}^{1} \frac{|\log \epsilon|^{-1}}{r} \, dr = \frac{1 + o(1)}{4\pi} \log \frac{1}{\epsilon}
\]
after some elementary computations. Combining these facts and recalling that \(|1 - r| \leq C |\log \epsilon|^{-1/2}\) in \(A_1\), we see that
\[
\frac{1}{|\log \epsilon|} \int_{A_1} r |U_c^i|^2 \, d\mathbf{r} = \frac{1}{4\pi} + o(1),
\]
as $\varepsilon \to 0$. It also follows that

$$\frac{1}{|\log \varepsilon|} \int_{A_1} r |U_i r|^2 \, dx = \frac{1}{2\pi} + o(1) \quad \text{as } \varepsilon \to 0.$$ 

On the other hand, we know from Steps 1 - 3 that

$$\frac{1}{|\log \varepsilon|} \int_{\mathcal{H}} r |U_i|^2 \, dx = \frac{1}{|\log \varepsilon|} \int_{\mathcal{H}} r |u|^2 \, dx + o(1) = \frac{1}{2\pi} + o(1)$$

(The last inequality is a consequence of (168), (170), (171).) Thus

$$\frac{1}{|\log \varepsilon|} \int_{\mathcal{H} \setminus A_1} |U_i|^2 \, dx = o(1).$$

Putting all these together, we conclude that

$$T_2 = \int_{A_1} r |U_i|^2 \, dx = \int_{A_1} \frac{r |U_i|^2}{|\log \varepsilon|} \, dx + \int_{A_1} \frac{r |U_i|^2}{|\log \varepsilon|} \, dx = \frac{1}{4\pi} + o(1)$$

as $\varepsilon \to 0$. \qed
CHAPTER 4

Point vortices in the Gross-Pitaevskii equation in 2 dimensions

In this chapter we will study the Gross-Pitaevskii equation

\[ i\partial_t \psi - \Delta \psi + \frac{1}{\varepsilon^2}(|\psi|^2 - 1)\psi = 0, \quad 0 < \varepsilon \ll 1 \tag{203} \]

for \( \psi : \mathcal{D} \times [0,T) \to \mathbb{C}, \) where \( \mathcal{D} \) is a bounded, open, connected and simply connected subset of \( \mathbb{R}^2, \) and that \( \partial \mathcal{D} \) is smooth. We will impose the boundary conditions

\[ \nu \cdot \nabla \psi = 0 \quad \text{on} \quad \partial \mathcal{D}. \tag{204} \]

We recall that some relevant notation is introduced in Section 4.

A solution \( \psi \) is interpreted as a wave function describing an ideal dilute quantum mechanical fluid. The following quantities are relevant to both the physical interpretation and the mathematical study of (203).

- **energy density** \( e_\varepsilon(\psi) := \frac{1}{2} |\nabla \psi|^2 + \frac{(|\psi|^2 - 1)^2}{4\varepsilon^2}. \)
- **mass density** \( |\psi|^2. \)
- **momentum density**, or current \( j(\psi) := (i\psi, \nabla \psi) = \text{vector with kth component} \ (i\psi, \partial_k \psi). \)
  
  If one writes \( \psi = \rho e^{i\phi}, \) then \( j(\psi) = \rho^2 \nabla \phi. \)
- The **vorticity** is defined by \( \omega(\psi) := \frac{1}{2} \nabla^\perp \cdot j(\psi). \) If we identify \( \psi = \psi_1 + i\psi_2 \) with the \( \mathbb{R}^2 \)-valued map \( (\psi_1, \psi_2), \) then \( \omega(\psi) \) is just the Jacobian determinant
  \[ \omega(\psi) = \det(\nabla \psi) = \frac{\partial(\psi_1, \psi_2)}{\partial(x_1, x_2)} = \partial_1 \psi_1 \partial_2 \psi_2 - \partial_2 \psi_1 \partial_1 \psi_2. \]

We will eventually prove a theorem very parallel to Theorem 4 showing that point vortices in solutions of (203) evolve by the point vortex ODEs, modified

1. **variational estimates**

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\(^1\)As remarked earlier, one difference between Euler and GP is that for GP, there is no analog of the pseudo-energy. As a result any function on all of \( \mathbb{R}^2 \) whose total vorticity is nonzero ends up having infinite energy. There are things one can do to get around this, but they introduce technical complications that we prefer to avoid. So a bounded domain is convenient.
CHAPTER 5

Appendices

1. Some general background material

1.1.  

1.2. Function spaces, including Sobolev inequalities. The $\mathcal{D}$ be an open subset of $\mathbb{R}^N$.

We assume familiarity with $L^p$ spaces.

We first introduce compact notation for derivatives of arbitrary order $k \geq 1$. We define

$$I(k, N) := \{ (\alpha_1, \ldots, \alpha_N) : \alpha_j \geq 0 \text{ for all } j, \alpha_1 + \ldots + \alpha_N = k \}.$$ 

Assume that $f : \mathcal{D} \to \mathbb{R}^k$ is rather smooth. For $\alpha \in I(k, N)$, we use the notation

$$D^\alpha f := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_N})^{\alpha_N} f.$$ 

Thus $D^\alpha f$ denotes a generic $k$th order partial derivative of the function $f$ of $n$ variables. The above definition certainly makes sense if $f$ is $C^k$. We will use it below for less smooth functions.

If $k \geq 1$ is an integer and $p \in [1, \infty)$, then

$$W^{k,p}(\mathcal{D}) := \{ f \in L^p(\mathcal{D}) : \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p} < \infty \}$$

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We first change variables in the definition (163) of $F$, letting $\phi = \alpha^2$ and using the identity $\cos \alpha = \cos^2 \phi - \sin^2 \phi$. This leads to

$$F(s) = 2 \int_0^{\pi/2} \frac{\cos^2 \phi - \sin^2 \phi}{\sqrt{s + 4 \sin^2 \phi}} d\phi = \int_0^{\pi/2} \frac{1 - 2 \sin^2 \phi}{\sqrt{\sigma^2 + \sin^2 \phi}} d\phi, \quad \sigma^2 = \frac{s}{4}. \tag{205}$$

The main term is

$$F_1(206) \int_0^{\pi/2} \frac{1}{\sqrt{\sigma^2 + \sin^2 \phi}} d\phi = \int_0^{\pi/2} \frac{\cos \phi}{\sqrt{\sigma^2 + \sin^2 \phi}} d\phi + \int_0^{\pi/2} \frac{1 - \cos \phi}{\sqrt{\sigma^2 + \sin^2 \phi}} d\phi \tag{206}$$

The first integral can be evaluated explicitly.

$$\int_0^{\pi/2} \frac{\cos \phi}{\sqrt{\sigma^2 + \sin^2 \phi}} d\phi = \int_0^1 \sqrt{\frac{t}{\sigma^2 + t^2}} = \left[ \log \left( t + \sqrt{\sigma^2 + t^2} \right) \right]_0^1 = \log \left( \frac{1 + \sqrt{\sigma^2 + t^2}}{\sigma} \right) = \frac{1}{2} \log \frac{4}{s} + \log 2 + O(\sigma^2) \text{ as } \sigma \to 0.$$  

Also, for $\sigma$ large is is clear that $\int_0^1 \frac{dt}{\sqrt{\sigma^2 + t^2}} \leq \log^+ \frac{1}{\sigma} + C$. For the other term in (206),

$$\int_0^{\pi/2} \frac{1 - \cos \phi}{\sqrt{\sigma^2 + \sin^2 \phi}} d\phi = \int_0^{\pi/2} \frac{1 - \cos \phi}{\sin \phi} d\phi + \int_0^{\pi/2} (1 - \cos \phi) \left( \frac{1}{\sqrt{\sigma^2 + \sin^2 \phi}} - \frac{1}{\sin \phi} \right) d\phi. \tag{207}$$

We explicitly evaluate

$$\int_0^{\pi/2} \frac{1 - \cos \phi}{\sin \phi} d\phi = -2 \log(\cos(\frac{\phi}{2}))\big|_0^{\pi/2} = \log 2.$$  

The other integral is negative. (One can also check that for small $\sigma$ it is bounded by $C\sigma^2 \log \frac{1}{\sigma}$.)

Next, it is clear that

$$-2 \leq -2 \int_0^{\pi/2} \frac{\sin^2 \phi}{\sqrt{\sigma^2 + \sin^2 \phi}} d\phi \leq 0 \text{ for all } \sigma.$$  

For $\sigma$ small we have the more precise estimate

$$-2 \int_0^{\pi/2} \frac{\sin^2 \phi}{\sqrt{\sigma^2 + \sin^2 \phi}} d\phi = -2 \int_0^{\pi/2} \frac{\sigma^2}{\sqrt{\sigma^2 + \sin^2 \phi}} d\phi + 2 \int_0^{\pi/2} \frac{\sigma^2}{\sqrt{\sigma^2 + \sin^2 \phi}} d\phi = -2 \int_0^{\pi/2} \sin d\phi + O(\sigma^2) = -2 + O(\sigma^2).$$  

It is also easy to see that