MAT234H1S, Spring 2003: some problems on first order linear systems.

These problems relate to material covered in lectures in the week of February 24-28, 2003.

(1) Let $A = \begin{pmatrix} 0 & 3 \\ 3 & -8 \end{pmatrix}$
   (a) Find the eigenvalues and eigenvectors of A.
   (b) Find e^A.
   (c) Find e^{tA}.
   (d) Give the solution of the first-order system $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(2) Let $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
   (a) Find the eigenvalues and eigenvectors of A.
   (b) Find e^A.
   (c) Find e^{tA}.
   (d) Give the solution of the first-order system $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(1) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

(3) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Compute e^{tA} directly from the definition of a matrix exponential. (This matrix is not diagonalizable, so e^{tA} cannot be found by the formula $e^{tA} = Be^D B^{-1}$, where $B^{-1}AB$ is a diagonal matrix D.)

(4) Let $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.
   (a) Find e^A.
   Hint: This matrix too is not diagonalizable. Use the fact, discussed in the lectures, that $e^A = e^{(A-\alpha I)}e^{\alpha I}$ for any real number α. Make a suitable choice of α to reduce this to the case of the previous exercise.
   (b) Find e^{tA}.
   (c) Find the solution of the first-order system $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(5) Consider the second order linear equation $ay'' + by' + cy = 0$
   (a) Rewrite this as a first-order system of the form $\mathbf{y}' = A\mathbf{y}$ (You can copy this from the class notes if you like, but it is easier just to do it.)
   (b) Find the equation that the eigenvalues of the matrix A solve.
   (c) If you solve the equation by the method of Chapter 3, without first converting it to a first-order system, you have to find the roots of the "characteristic polynomial". Find the characteristic polynomial for this equation.
   (d) The point, in case you missed it, is that your answers to parts (b) and (c) should be the same.

(6) Show that if $\mathbf{v}$ is an eigenvector for a matrix A, with eigenvalue λ (that is, if $A\mathbf{v} = \lambda\mathbf{v}$) then $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$ is a solution of the first-order system $\mathbf{y}' = A\mathbf{y}$.

(7) Consider the second-order system

$$m_1 y_1'' = -k_1 y_1 + k_2(y_2 - y_1)$$
$$m_2 y_2'' = -k_3 y_2 - k_2(y_2 - y_1).$$

This describes the positions of a system of two masses and three springs, without friction.

(a) Draw a picture indicating the configuration of masses and springs described by the above system.

Now for simplicity let's assume that $m_1 = m_2 = 1$ and that $k_1 = k_3 = 1$, and $k_2 = 2$, so that the system becomes

$$y_1'' = -3y_1 + 2y_2$$
$$y_2'' = 2y_1 - 3y_2.$$

(b) Write this as a first-order system of two equations, of the form

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \end{pmatrix},$$

for a suitable matrix A, where v_1 and v_2 denote the velocity of the first and second masses respectively.

(c) Find the eigenvalues and eigenvectors of A. (Hint: if $\mathbf{v}$ is a complex eigenvector for a complex eigenvalue λ, then $\bar{\mathbf{v}}$ is an eigenvector for the eigenvalue $\bar{\lambda}$, where the bar denotes complex conjugate. This cuts in half the amount of calculation you need to do.)

(d) Find four linearly independent complex vector-valued solutions of the first-order system of equations, that is, solutions which are vectors with complex entries. (Hint: use the previous exercise.)

(e) By taking real and imaginary parts of the complex solutions, find four linearly independent real solutions of the system of equations.

(f) Think geometrically about the solutions you have found. How do they behave? What is the relationship between the motion of the two masses?