1. Prove that if $V \in L^p(\mathbb{R}^d)$ for some $p < d$ then for any $\phi \in C_0^\infty(\mathbb{R}^d)$,
\[
\int_1^\infty \|Ve^{iH_0t/\hbar}\phi\|_{L^2(\mathbb{R}^d)}dt < \infty.
\]
Here $H_0 = -\hbar^2/2m\Delta$, with domain chosen to make $H_0$ self-adjoint.

**Hint:** use the result of Problem 2 below, together with properties of the operator $e^{itH_0/\hbar}$ that we have already established.

Following arguments given in the book, the result of this exercise implies the existence of the wave operator $\Omega^+ = \text{s-lim}_{t \to \infty} e^{iHt/\hbar} e^{-iH_0t/\hbar}$ for a Schrödinger operator $H$ with potential $V \in L^p(\mathbb{R}^d)$ for some $p < d$. This applies in particular if $V(x) \leq C(1 + |x|)^{-\mu}$ for some $\mu > 1$.

2. Prove that if $f : \mathbb{R}^d \to \mathbb{C}$ belongs to both $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$ for some $1 \leq p < q \leq \infty$, then $f \in L^r(\mathbb{R}^d)$ for every $p < r < q$, and
\[
\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta} \|f\|_{L^q}^{1-\theta}
\]
for $\theta$ such that $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$.

3. GS problem 7.1. More precisely, using notation as in GS:

Determine the spectrum of the operator $x_j$ for some $j \in \{1, 2, 3\}$ with domain $D(x_j) = \{\psi \in L^2(W) : \psi(x) = 0 \text{ when } x \in \partial W\}$ and the eigenvalues of $p_j = -i\hbar\frac{\partial}{\partial x_j}$ with domain $D(p_j) = \{\psi \in L^2(W) : p_j \psi \in L^2(W), \psi(x) = 0 \text{ when } x \in \partial W\}$.

4. GS problem 7.2. More precisely, using notation as in GS:

Determine the spectrum of the operator $x_j$ for some $j \in \{1, 2, 3\}$ with domain $D(x_j) = \{\psi \in L^2(W) : \psi(x) \text{ periodic }\}$ and the eigenvalues of $p_j = -i\hbar\frac{\partial}{\partial x_j}$ with domain $D(p_j) = \{\psi \in L^2(W) : p_j \psi = L^2(W), \psi \text{ periodic }\}$.

5. What is strange about questions 4 and 5?