1. Suppose that $A$ is a self-adjoint operator with spectral family $\{E_\lambda\}$, and that $\psi$ is an element of the Hilbert space such that $\langle \psi, \phi \rangle = 0$ for every eigenvector $\phi$ of $E$.

(a) Prove that $\lambda \mapsto \langle E_\lambda \psi, \psi \rangle$ is continuous.

(b) Prove that $\lambda \mapsto \langle E_\lambda \psi, \eta \rangle$ is continuous for every fixed $\eta \in \mathcal{H}$.

2. We have proved the following theorem in the lectures:

**Theorem 1.** If $K$ is a rank-1 operator and $\phi \perp \mathcal{H}_e$, then

$$
\frac{1}{T} \int_0^T \| Ke^{itA} \phi \|^2 \, dt \to 0 \quad \text{as } T \to \infty.
$$

Moreover

$$
\left\| \frac{1}{T} \int_0^T e^{-itA} K P_e e^{itA} \, dt \right\| \to 0 \quad \text{as } T \to \infty.
$$

The integral in (2) is understood to be a s-lim of Riemann sums.

Here we use the notation

$$
\mathcal{H}_e := \text{the closed span of the eigenvectors of } A
$$

$$
= \text{the smallest closed subspace containing all eigenvectors}.
$$

We write $P_e$ to denote orthogonal projection onto $\mathcal{H}_e$ and $P_e^\perp := I - P_e$.

Prove that the theorem is still valid if $K$ is a compact operator. In doing this, you can take for granted the following

**Lemma 1.** Every compact operator is a norm limit of finite-rank operators. In other words, if $K$ is compact and $\varepsilon > 0$, then there exists an operator $K_\varepsilon$ which is a linear combination of finitely many rank-1 operators, and such that $\|K - K_\varepsilon\| < \varepsilon$.

3. Suppose that $A$ is an unbounded, self-adjoint operator on a Hilbert space $\mathcal{H}$. For $\psi, \phi \in D(A)$, define

$$
[\psi, \phi] := \langle A\psi, A\phi \rangle + \langle \psi, \phi \rangle.
$$

Verify that $[\cdot, \cdot]$ is a norm, and that with this norm, $D(A)$ is a Hilbert space.