Therefore, *if ϕ is a Borel regular measure over a metric space, the statements* (1) *and* (2) *of* 2.2.2 *hold for every ϕ measurable set B.*

If ϕ is Borel regular and A is a Borel set, then $\phi \llcorner A$ is Borel regular.

We also observe that if ϕ is any measure over a topological space X such that all Borel subsets of X are ϕ measurable, and if

$$\psi(A) = \inf\{\phi(B)\colon\ A \subset B \text{ and } B \text{ is a Borel set}\}$$

whenever $A \subset X$, then ψ is a Borel regular measure, and $\psi(A) = \phi(A)$ in case A is a Borel set.

2.2.4. Theorem. *If ϕ is a Borel regular measure over a complete, separable metric space X, $0 < \phi(A) < \infty$, and $\phi(\{x\}) = 0$ whenever $x \in A$, then A has a ϕ nonmeasurable subset.*

Proof. We consider the class Γ of all closed subsets C of A for which $\phi(C) > 0$, hence $\operatorname{card}(C) = 2^{\aleph_0}$. Noting that $\operatorname{card}(\Gamma) \leq 2^{\aleph_0}$, we wellorder Γ so that, for each $C \in \Gamma$, the set Γ_C of all predecessors of C has cardinal less than $2^{\aleph_0}$. By induction with respect to this wellordering we define functions f and g on Γ such that, for each $C \in \Gamma$, $f(C)$ and $g(C)$ are distinct elements of

$$C \sim [f(\Gamma_C) \cup g(\Gamma_C)];$$

this is possible because

$$\operatorname{card}[f(\Gamma_C) \cup g(\Gamma_C)] = 2\operatorname{card}(\Gamma_C) < 2^{\aleph_0} = \operatorname{card}(C).$$

Since both $\operatorname{im} f$ and $A \sim \operatorname{im} f$ (which contains im g) meet every member of Γ, neither set contains any member of Γ. If these two sets were ϕ measurable, both would have ϕ measure 0, hence $\phi(A) = 0$. Therefore either A or $\operatorname{im} f$ is ϕ nonmeasurable.

2.2.5. By a **Radon measure** we mean a measure ϕ, over a *locally compact Hausdorff space X*, with the following three properties:

If K is a compact subset of X, then $\phi(K) < \infty$.

If V is an open subset of X, then V is ϕ measurable and

$$\phi(V) = \sup\{\phi(K)\colon\ K \text{ is compact, } K \subset V\}.$$

If A is any subset of X, then

$$\phi(A) = \inf\{\phi(V)\colon\ V \text{ is open, } A \subset V\}.$$

We observe that, in case ϕ is a Radon measure, then

$$\phi(X \sim \operatorname{spt} \phi) = 0,$$