1. Suppose that $U$ is a bounded open subset of $\mathbb{R}^n$, and suppose that $1 \leq p < n$. Give an example of a sequence of functions $\{u_k\} \subset W^{1,p}(U)$ such that 

$$
\|u_k\|_{W^{1,p}(U)} \leq C \quad \text{but no subsequence converges to any limit in } L^{p^*}(U).
$$

where $p^*$ is the Sobolev exponent $p^* = \frac{np}{n-p}$.

(Hint: look for a sequence such that $u_k \rightarrow 0$ a.e. but $\|u_k\|_{L^{p^*}}$ is a constant independent of $k$.)

2. Give an example of a sequence of radial functions $u_k \in W^{1,p}(\mathbb{R}^n)$ for $1 \leq p < n$ such that 

$$
\|u_k\|_{W^{1,p}(U)} \leq C \quad \text{but no subsequence converges to any limit in } L^p(\mathbb{R}^n).
$$

(Similar hint to above.)

3. Let $U$ be the unit ball in $\mathbb{R}^n$, and let $p$ be a number such that $1 \leq p < \infty$. Show by example that there exists a sequence of functions $\{u_k\} \subset W^{1,p}(U)$ such that

(a) $\frac{\partial u_k}{\partial \nu} = 0$ on $\partial U$ (in particular, each $u_k$ should be smooth enough that $\frac{\partial u_k}{\partial \nu}$ is well-defined)

(b) The sequence $u_k$ converges in $W^{1,p}(U)$ to a limit $u$; and

(c) $\frac{\partial u}{\partial \nu}$ is well-defined and does not identically vanish on $\partial U$.

This shows that one cannot hope to minimize any functional in a set of the form $A := \{u \in H^1(U) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U \}$. Indeed, we do not know how to make sense of $\frac{\partial u}{\partial \nu}$ for a general function $u \in H^1(U)$, since we have not established anything analogous to the trace theorem for $\frac{\partial u}{\partial \nu}$.

If I ask this question on a quiz, I will only ask you to write down the answer, but not to check in detail that the examples you give have the desired properties. (But you should write down a correct answer!)

4. Prove the following Poincaré type inequality:

If $U \subset \mathbb{R}^n$ is a bounded open set with smooth boundary, then there exists a constant $C$ (depending on the domain $U$) such that

$$
\|u\|_{L^2(U)} \leq C \left( \|Du\|_{L^2(U)} + \|Tu\|_{L^2(\partial U)} \right)
$$

where $Tu \in L^2(\partial U)$ denotes the trace of $u$.

(This was needed in a previous homework problem. In your solution, feel free to write $u$ instead of $Tu$ for simplicity, when referring to the boundary values of $u$.)

5. Suppose that $H$ and $\tilde{H}$ are Hilbert spaces, and that $A : H \rightarrow \tilde{H}$ is a bounded linear operator. Prove that if $u_k \rightharpoonup u$ weakly in $H$, then $Au_k \rightharpoonup Au$ weakly in $\tilde{H}$.

(This implies in particular that if $u_k \rightharpoonup u$ weakly in $H^1(U)$ then $Tu_k \rightharpoonup Tu$ weakly in $L^2(\partial U)$, where $Tw$ denotes the trace of $w$. This also was needed for a previous homework problem. For this, you may need to recall the definition of the adjoint $A^*$ of a bounded linear operator $A : H \rightarrow \tilde{H}$.)
