PDEII, possible quiz questions,spring 2008,

These should be questions whose answers can be written in just a few lines, if you write concisely (which I encourage.)

For the first few questions, let U be a bounded, open subset of $\mathbb{R}^n$ with C^1 boundary, and for $1 < p \leq \infty$, define as usual $W^{-1,p}(U)$ to be the dual space of $W_0^{1,q}(U)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

- For $k \geq 1$, let $f_k(x_1, \ldots, x_n) = \sin(kx_1)$. Prove that

$$\|f\|_{H^{-1}(U)} \leq C/k$$

  for a constant C depending only on the domain U.

- Prove that if $p < n$ then $L^p \subset W^{-1,p^*}$ where $p^* = \frac{np}{n-p}$.

For the next question we need the concept of a "delta function at a point a", where $a \in U$. This "function", denoted δ_a, is actually not a function; rather, it is by definition the measure on U characterized by

$$\delta_a(A) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{if not} \end{cases} \qquad \text{for } A \subset U.$$

Thus, if f is a continuous function on U, we can integrate to find that

$$\int f(x)\delta_a(dx) = f(a).$$

- Show that for δ_a as above, $\delta_a \in W^{-1,p}(U)$ if $p < 1^* = \frac{n}{n-1}$.
- Show that if $1 < p < q$ then $W^{-1,q}(U) \subset W^{-1,p}(U)$. (This means that every element of $W^{-1,q}(U)$ can be identified with an element of $W^{-1,p}(U)$. This is not in general true on unbounded domains.)
- Evans, Chapter 6, problem 7.
- Evans, Chapter 6, problem 8.