PDEII, possible quiz questions, spring 2008,

These should be questions whose answers can be written in just a few lines, if you write concisely (which I encourage.)

For the first few questions, let $U$ be a bounded, open subset of $\mathbb{R}^n$ with $C^1$ boundary, and for $1 < p \leq \infty$, define as usual $W^{-1,p}(U)$ to be the dual space of $W_0^{1,q}(U)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

- For $k \geq 1$, let $f_k(x_1, \ldots, x_n) = \sin(kx_1)$. Prove that
  \[ \|f\|_{H^{-1}(U)} \leq C/k \]
  for a constant $C$ depending only on the domain $U$.

- Prove that if $p < n$ then $L^p \subset W^{-1,p^*}$ where $p^* = \frac{np}{n-p}$.

For the next question we need the concept of a “delta function at a point $a$”, where $a \in U$. This “function”, denoted $\delta_a$, is actually not a function; rather, it is by definition the measure on $U$ characterized by

\[
\delta_a(A) = \begin{cases} 
1 & \text{if } a \in A \\
0 & \text{if } a \notin A
\end{cases} \quad \text{for } A \subset U.
\]

Thus, if $f$ is a continuous function on $U$, we can integrate to find that

\[ \int f(x)\delta_a(dx) = f(a). \]

- Show that for $\delta_a$ as above, $\delta_a \in W^{-1,p}(U)$ if $p < 1^* = \frac{n}{n-1}$.

- Show that if $1 < p < q$ then $W^{-1,q}(U) \subset W^{-1,p}(U)$. (This means that every element of $W^{-1,q}(U)$ can be identified with an element of $W^{-1,p}(U)$. This is not in general true on unbounded domains.)

- Evans, Chapter 6, problem 7.

- Evans, Chapter 6, problem 8.