MAT 1061: notes on Nonlinear Ground States

We study solutions[1] $u \in H^1(\mathbb{R}^n)$ of the equation

$$-\Delta u + \lambda u - |u|^{p-2}u = 0 \qquad \text{in } \mathbb{R}^n \tag{1}$$

We will restrict our attention to dimensions $n \geq 3$; similar results hold in dimensions $n = 1, 2$, with small modifications.

We will study (1) using variational methods. Thus we define the functional

$$I[u] := \int_{\mathbb{R}^n} \frac{|Du|^2}{2} + \lambda \frac{|u|^2}{2} - \frac{|u|^p}{p}\ dx. \tag{2}$$

It will be useful to give a name to each of the 3 terms that make up I. Thus we use the notation I_D for the Dirichlet energy,

$$I_D[u] = \int_{\mathbb{R}^n} \frac{|Du|^2}{2}$$

and for $q \geq 1$ we write

$$I_q[u] := \int_{\mathbb{R}^n} \frac{|u|^q}{q}\ dx$$

so that $I = I_D + \lambda I_2 - I_p$.

We will first establish existence of solutions of (1) by a variational argument. We will then go on to establish some interesting properties of the specific solutions found by our variational technique.

1. EXISTENCE

In this section we prove

Theorem 1. *For $\Lambda > 0$, define*

$$\mathcal{A}_\Lambda \ := \ \{v \in H^1(\mathbb{R}^n) \ : \ \lambda I_2[v] - I_p[v] = -\Lambda\}.$$

Then for every $\Lambda > 0$, there exists $u \in \mathcal{A}_\Lambda$ such that u is positive and radial, and

$$I_D[u] = \min_{v \in \mathcal{A}_\Lambda} I_D[v].$$

Moreover, there exists a number Λ_* for which every minimizer of I_D in $\mathcal{A}_{\Lambda_*}$ is a solution of (1) *(including in particular the minimizer found above.)*

To say that u is radial means that there exists some function $\tilde{u} : [0, \infty) \to [0, \infty)$ such that $u(x) = \tilde{u}(|x|)$ a.e..

[1]It turns out that every weak solution is in fact C^∞, so we will mostly not distinguish carefully between solutions and weak solutions, except when it is necessary.

1.1. **some lemmas.** We will need the following deep fact from analysis, which we will accept without proof:

Proposition 1 (Symmetrization decreases the Dirichlet energy)**.** *Suppose that* $u \in L^p(\mathbb{R}^n;\mathbb{C})$ *for some* p*, and that* $Du \in L^2(\mathbb{R}^n)$*. Define the* Schwarz symmetrization *of* u *to be the unique nonnegative, radial function in* $L^2(\mathbb{R}^n)$*, denoted* u^**, with the property that*

$$\mathcal{L}^n\left(\{x \in \mathbb{R}^n \ : \ u^*(x) \geq \lambda\}\right) \ = \ \mathcal{L}^n\left(\{x \in \mathbb{R}^n \ : \ u(x) \geq \lambda\}\right)$$

for every $\lambda \in \mathbb{R}$*, and as a result*

$$\|u^*\|_{L^q(\mathbb{R}^n)} = \|u\|_{L^q(\mathbb{R}^n)}$$

for every $q \in [1,\infty]$ *(where it is possible that both sides equal* $+\infty$ *for some* q*). Then* $Du^* \in L^2$*, and*

$$\int_{\mathbb{R}^n} |Du^*|^2 \ dx \ \leq \ \int_{\mathbb{R}^n} |Du|^2 \ dx. \tag{3}$$

Remarks about the proof. □

Next, we have the following

Lemma 1. *Let* $H^1_{rad}(\mathbb{R}^n) := \{v \in H^1(\mathbb{R}^n) \ : \ v \text{ is radial}\}$*. Then* $H^1_{rad}(\mathbb{R}^n)$ *is compactly embedded in* $L^p(\mathbb{R}^n)$ *for every* $p \in (2, 2^*)$*.*

The lemma is interesting because Rellich's compactness theorem fails on $\mathbb{R}^n$. For example, if $u \in H^1(\mathbb{R}^n)$ is any nonzero function and x_k is any sequence of points tending to ∞, then the sequence $u_k(x) := u(x - x_k)$ is bounded in $H^1(\mathbb{R}^n)$ but does not have any convergent sequence in $L^p(\mathbb{R}^n)$ for any p.

Proof. **1**. We first claim that if $v \in H^1_{rad}(\mathbb{R}^n)$, then

$$|x|^{\frac{n-1}{2}}|v(x)| \ \leq C\|v\|_{L^2} \ \|Dv\|_{L^2} \qquad \text{for } a.e. x \in \mathbb{R}^n.$$

To prove this, note if $v \in H^1_{rad}(\mathbb{R}^n)$ is smooth with compact support, then

$$\begin{aligned} r^{n-1}|v(r)|^2 &= -\int_r^\infty \frac{d}{ds}(s^{n-1}|v(s)|^2)\, ds \\ &\leq -2\int_r^\infty s^{n-1} v(s)\, v'(s)\, ds \\ &\leq C\left(\int_r^\infty s^{n-1}|v(s)|^2\, ds\right)^{1/2}\left(\int_r^\infty s^{n-1}|v'(s)|^2\, ds\right)^{1/2} \\ &\leq C\|v\|_{L^2}\|Dv\|_{L^2}. \end{aligned}$$

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$, it follows that the claim holds for all $v \in H^1_{rad}$.

2. Now suppose that $\{u_k\} \subset H^1(\mathbb{R}^n)$ is a sequence such that $\|u_k\|_{H^1} \leq C$. After passing to a subsequence, we may assume that there exists some $u \in H^1(\mathbb{R}^n)$ such that

$$u_k \rightharpoonup u \ \text{ weakly in } L^2(\mathbb{R}^n), \qquad Du_k \rightharpoonup Du \ \text{ weakly in } L^2(\mathbb{R}^n).$$

Note that $\|u\|_{H^1} \leq \sup_k \|u_k\|_{H^1} \leq C$ To complete the proof we will show that for $p \in (2, 2^*)$ and arbitrary $\varepsilon > 0$, there exists some $\ell = \ell(\varepsilon)$ such that $\|u_k - u\|_{L^p} < \varepsilon$ for all $k \geq \ell$. Then for R to be chosen,

$$\begin{aligned}\|u-u_k\|_{L^p} &= \|u-u_k\|_{L^p(B_R)} + \|u-u_k\|_{L^p(\mathbb{R}^n\setminus B_R)} \\ &\leq \|u-u_k\|_{L^p(B_R)} + \|u-u_k\|^{2/p}_{L^2(\mathbb{R}^n\setminus B_R)}\|u-u_k\|^{(p-2)/p}_{L^\infty(\mathbb{R}^n\setminus B_R)}\end{aligned}$$

Clearly $\|u-u_k\|^{2/p}_{L^2(\mathbb{R}^n\setminus B_R)} \leq C$, and in view of Step 1, if $v \in H^1_{rad}$ then

$$\|v\|_{L^\infty(\mathbb{R}^n\setminus B_R)} \leq CR^{\frac{1-n}{2}}\|v\|_{H^1}$$

Thus by taking R large enough, we can arrange that

$$\|u-u_k\|^{2/p}_{L^2(\mathbb{R}^n\setminus B_R)}\|u-u_k\|^{(p-2)/p}_{L^\infty(\mathbb{R}^n\setminus B_R)} \leq \frac{\varepsilon}{2}.$$

Having fixed R, we note that since B_R is bounded, Rellich's Compactness Theorem and the weak convergence $u_k \rightharpoonup u$ imply that $u_k \to u$ in $L^p(B_R)$ for $p < 2^*$. Thus there exists some ℓ such that $\|u-u_k\|_{L^p(B_R)} < \frac{\varepsilon}{2}$ if $k \geq \ell$. □

1.2. **proof of Theorem 1.** Using the above results, we present the proof of the theorem.

proof of Theorem 1. **1**. Fix $\Lambda > 0$, and let $\{v_k\} \subset \mathcal{A}_\Lambda$ be a sequence such that

$$I_D[v_k] \to \inf_{\mathcal{A}_\Lambda} I_D.$$

Let u_k denote the Schwarz symmetrization v_k^* of v_k.

It is clear that $u_k \in \mathcal{A}_\Lambda$, and it follows from Proposition 1 that $I_D[u_k] \to \inf_{\mathcal{A}_\Lambda} I_D$.

Next, note that for every $v \in \mathcal{A}_\Lambda$,

$$\frac{\lambda}{2}\|v\|_2^2 \quad = \quad \frac{1}{p}\|v\|_p^p - \Lambda \quad \leq \quad \frac{1}{p}\|v\|_p^p \quad \leq \quad \frac{1}{p}\left(\|v\|_2^\theta\ \|Dv\|_2^{1-\theta}\right)^p$$

for θ as defined in (7), see Lemma 2 below. Upon rearranging we find that

$$\|v\|_2^{2-\theta p} \leq C\|Dv\|_2^{(1-\theta)p}.$$

The definition of θ implies that $2-\theta p = 2p\frac{1-\theta}{2^*} > 0$, so it follows that $\|v\|_2 \leq C\|Dv\|_2^\alpha$ for some $\alpha > 0$, for $v \in \mathcal{A}_\Lambda$.

It follows that $\{u_k\}$ is bounded in $H^1(\mathbb{R}^n)$.

2. Next, in view of Lemma 1, we can pass to a subsequence (still labelled $\{u_k\}$) such that

(4)
$$u_k \rightharpoonup u \ \text{ weakly in } L^2, \quad Du_k \rightharpoonup Du \ \text{ weakly in } L^2, \text{ and } \quad u_k \to u \ \textit{strongly} \text{ in } L^p.$$

Upon passing to a further subsequence if necessary, we can also assume that $u_k \to u$ a.e. in $\mathbb{R}^n$. Each u_k is nonnegative and radial, so the same properties are inherited by u.

3. We next show that $u \in \mathcal{A}_\Lambda$ It follows from (4) that

$$\int |u|^2 \leq \liminf_k \int |u_k|^2$$
$$\int |Du|^2 \leq \liminf_k \int |Du_k|^2 = \inf_{\mathcal{A}_\Lambda} I_D$$
$$\int |u|^p = \lim_k \int |u_k|^p.$$

Since each u_k belongs to $\mathcal{A}_\Lambda$, we deduce that

$$-\lambda I_2[u] + I_p[u] \geq \liminf_k \left(-\lambda I_2[u_k] + I_p[u_k]\right) = \Lambda$$

Suppose toward a contradiction that strict inequality holds. Then there exists some $a \in (0,1)$ such that $au \in \mathcal{A}_\Lambda$, and clearly $I_D[au] = a^2 I_D[u] < \inf_{\mathcal{A}_\Lambda} I_D$, which is impossible. Thus $u \in \mathcal{A}_\Lambda$.

We conclude that u minimizes I_D in $\mathcal{A}_\Lambda$, as required.

4. next, we know that the Euler-Lagrange equation for the constrained variational problem satisfied by u is

$$-\Delta u = \alpha(-\lambda u + |u|^{p-2}u)$$

where α is a Lagrange multiplier. Then Lemma 3 proved below (see in particular (20)) implies that there exists a constant $c_{n,p}$ depending only on n and p, such that

$$\int |u|^p dx \ = \ c_{n,p}\lambda \int |u|^2 dx.$$

Since $u \in \mathcal{A}_\Lambda$, it follows that

$$\Lambda = -\frac{\lambda}{2}\int |u|^2 + \frac{1}{p}\int |u|^p = \left(\frac{c_{n,p}}{p} - \frac{1}{2}\right)\lambda \int |u|^2 \ = \ \left(\frac{1}{p} - \frac{1}{2c_{n,p}}\right)\int |u|^p. \tag{5}$$

In addition, Lemma 3 also implies that

$$\alpha = \frac{\int |Du|^2}{\int(-\lambda|u|^2 + |u|^p)}.$$

Then it follows from (5) that the Lagrange multiplier α depends only on Λ (even if there is more than one minimizer in $\mathcal{A}_\Lambda$). Thus we will write $\alpha(\Lambda)$ to indicate the unique Lagrange multiplier associated with minimizing in $\mathcal{A}_\Lambda$.

Finally, we wish to adjust Λ so that the corersponding Lagrange mutliplier $\alpha(\Lambda)$ equals 1. To do this, let us define a map $\delta_a : H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ by $\delta_a u(x) = u(ax)$, $a > 0$. Then by a change of variables,

$$\|\delta_a u\|_{L^q}^q = a^{-n}\|u\|_{L^q}^q \qquad I_D[\delta_a u] = a^{2-n} I_D[u]$$

for every $u \in H^1(\mathbb{R}^n)$ and every $a > 0$. It is clear that δ_a is invertible, and it follows that δ_a is a bijection of $\mathcal{A}_\Lambda$ onto $\mathcal{A}_{a^{-n}\Lambda}$, and that $\inf_{\mathcal{A}_{a^{-n}\Lambda}} I_D = \alpha^{2-n}\inf_{\mathcal{A}_\Lambda} I_D$. We conclude that

$$\alpha(a^{-n}\Lambda) = a^2\alpha(\Lambda).$$

This implies that there is a unique Λ_* such that $\alpha(\Lambda_*) = 1$, which completes the proof. $\square$

2. A DIFFERENT EXISTENCE PROOF

In this section we give an alternative argument in which we again find solutions of (1) via a different variational problem.

First we recall

Lemma 2. *If $v \in W^{1,q}(\mathbb{R}^n)$ for $q < n$, then $v \in L^s(\mathbb{R}^n)$ for all $s \in (q, q^*)$, and in addition*

$$\|v\|_{L^s(\mathbb{R}^n)} \ \le \ C\|v\|^{\theta}_{L^q(\mathbb{R}^n)} \ \|Dv\|^{1-\theta}_{L^q(\mathbb{R}^n)} \tag{6}$$

where θ is defined by

$$\frac{1}{s} = \frac{\theta}{q} + \frac{1-\theta}{q^*}. \tag{7}$$

The proof is an exercise. Inequality (6) is sometimes called the Sobolev-Nirenberg-Gagliardo inequality. It implies that if we define

$$G[v] := \frac{\|v\|^{\theta}_{L^2(\mathbb{R}^n)} \ \|Dv\|^{1-\theta}_{L^2(\mathbb{R}^n)}}{\|v\|_{L^p(\mathbb{R}^n)}}$$

then $G[v] \ge C^{-1}$ for every $v \in H^1(\mathbb{R}^n) \setminus \{0\}$. (Here p is the same number appearing in (1), so that $2 < p < 2^*$, and θ is as in Lemma 6.)

The next theorem shows that at least some solutions of (1) minimize $G[\cdot]$, which means that they are extremal for the Sobolev-Nirenberg-Gagliardo inequality.

Theorem 2. *There exists a nonnegative radial function $u \in H^1(\mathbb{R}^n)$ such that*

$$G[u] \le G[v] \quad \text{for all } v \in H^1(\mathbb{R}^n) \setminus \{0\}. \tag{8}$$

Moreover, if u is any minimizer of G, then

$$-\alpha\Delta u + \beta u - \gamma|u|^{p-2}u = 0 \tag{9}$$

for positive constants α, β, γ. Finally, there exists a minimizer of G that solves (1).

Proof. **1**. First, it is easy to check that for any $v \in H^1(\mathbb{R}^n)$ and any $a, b > 0$, if we define $v_{a,b}(x) := bv(ax)$, then $G[v_{a,b}] = G[v]$. Indeed, this follows from noting that

$$\|v_{a,b}\|_{L^q} = ba^{-\frac{n}{q}}\|v\|_{L^q} \quad \text{for every } q, \qquad \|Dv_{a,b}\|_{L^2} = ba^{1-\frac{n}{2}}\|Dv\|_{L^2(\mathbb{R}^n)},$$

together with some arithmetic. These identities also imply that, given any nonzero $v \in H^1(\mathbb{R}^n)$, we can choose a, b such that $\|v_{a,b}\|_{L^2} = \|Dv_{a,b}\|_{L^2} = 1$.

2. let $\{v_k\} \subset H^1(\mathbb{R}^n) \setminus \{0\}$ be a sequence such that

$$G[v_k] \to \inf_{H^1\setminus\{0\}} G.$$

For each k, we let $u_k := (v_k^*)_{a,b}$, for a, b chosen so that $\|u_k\|_{L^2} = \|Du_k\|_{L^2} = 1$. (That is, u_k is obtained from v_k by first symmetrizing as in Proposition 1, and then by rescaling as in Step 1 above.) It follows that $G[u_k] = G[v_k^*] \le G[v_k]$, so that $\{u_k\}$ is still a minimizing sequence.

Then owing to Lemma 1, we can pass to a subsequence (still labelled $\{u_k\}$) such that
(10)
$$u_k \rightharpoonup u \text{ weakly in } L^2, \quad Du_k \rightharpoonup Du \text{ weakly in } L^2, \text{ and } \quad u_k \to u \textit{ strongly} \text{ in } L^p,$$

for every $p \in (2, 2^*)$. Upon passing to a further subsequence if necessary, we can also assume that $u_k \to u$ a.e. in $\mathbb{R}^n$. Each u_k is nonnegative and radial, so the same properties are inherited by u.

Because of (10), standard weak lowersemicontinuity arguments imply that (8) holds.

3. A calculation reveals that (9) is the Euler-Lagrange equation for G, with

$$\alpha = \frac{1-\theta}{\|Du\|_{L^2}^2},\ \beta = \frac{\theta}{\|u\|_{L^2}^2},\ \gamma = \frac{1}{\|u\|_{L^p}^p}.$$

4. It follows from Step 1 that if u is a minimizer, then so is $u_{a,b}$. Thus, starting with the minimizer found in Step 2 above, we can select a, b so for the associated constants α, β, γ, one has $\beta/\alpha = \lambda$ and $\gamma/\alpha = 1$. This is a minimizer of G that is also a solution of (1). □

3. PROPERTIES OF SOLUTIONS

In this section we give some further properties of solutions of (1) and related equations, some with proofs and some without.

Proposition 2 (regularity and decay)**.** *Assume that $u \in H^1(\mathbb{R}^n)$ is a weak solution of*

$$-\Delta u + au - b|u|^{p-2}u = 0 \qquad \textit{in } \mathbb{R}^n. \tag{11}$$

Then:

a. *For every $q \in [2, \infty)$, there exists a constant C, depending on n, p, q and $\|u\|_{H^1}$, such that*

$$u \in W^{3,q}(\mathbb{R}^n), \qquad \textit{and } \|u\|_{W^{3,q}} \leq C.$$

In particular, $|D^\alpha u| \to 0$ uniformly as $|x| \to \infty$ for all $|\alpha| \leq 2$.

b. *There exist $C, \varepsilon > 0$ such that*

$$e^{\varepsilon|x|}\left(|u(x)| + |Du(x)|\right) \leq C$$

for all $x \in \mathbb{R}^n$.

The proof of Proposition 2 uses the Calderon-Zygmund estimates. The version of these estimates that is most convenient for our purposes states that for $1 < p < \infty$ and $a > 0$, there exists a constant C (depending on p, a and n, such that if $u \in H^1(\mathbb{R}^n)$ and $-\Delta u + au = f \in L^q(\mathbb{R}^n)$, then

$$u \in W^{2,q}(\mathbb{R}^n), \qquad \text{and } \|u\|_{W^{2,q}} \leq C\|f\|_{L^q}. \tag{12}$$

We assume this result, and we sketch the *proof of Proposition 2.* **1**. Let u solve (11).

Recall that $2 < p < 2^*$, so that $1 < 2^*/(2^*-1) < 2^*/(p-1) < 2^*$. By the Sobolev embedding theorem

$$\||u|^{p-2}u\|_{2^*/(p-1)} \;=\; \|u\|_{2^*}^{p-1} \;\le\; C\|u\|_{H^1}^{p-1}.$$

Then (11) and the Calderon-Zygmund estimate (12) imply that

$$u \in W^{2,2^*/(p-1)}, \text{ with } \quad \|u\|_{W^{2,2^*/(p-1)}} \le C\|u\|_{H^1}^{p-1}. \tag{13}$$

Since $2^*/(p-1) > 2$, we have gained regularity.

Note that if we know that $u \in W^{2,q}$ for $q < n/2$, then almost exactly the same argument, using the Sobolev embedding $W^{2,q} \hookrightarrow L^{q^{**}}$ and the Calderon-Zygmund estimate, implies that

$$u \in W^{2,q^{**}/(p-1)}, \text{ with } \|u\|_{W^{2,q^{**}/(p-1)}} \le C\|u\|_{W^{2,q}}^{p-1} \quad . \tag{14}$$

Here $q^{**} = (q^*)^*$, where $(\ )^*$ denotes the Sobolev exponent for $(\ )$, so that $\frac{1}{q^{**}} = \frac{1}{q} - \frac{2}{n}$, or equivalently $q^{**} = \frac{nq}{n-2q}$. One can easily check that if $q \ge q_0 = 2^*/(p-1)$ then there exists some $\alpha > 1$ (depending on p and n) such that $q^{**}/(p-1) > \alpha q$. Thus by starting from (13) and iterating (14) a finite number of times, one finds that $u \in W^{2,q}$ for some $q > n/2$, which implies that $u \in L^\infty$. Then $|u|^{p-1}u$ belongs to L^q for every $q \in [2,\infty)$, and (12) implies that $u \in W^{2,q}$ for every $q \in [2,\infty)$.

2. To get higher regularity, we argue as follows: multiply (11) by v_{x_i} for some $v \in H^1(\mathbb{R}^n)$ and some i. Then after integrating by parts and using Step 1, we find that $w := u_{x_i}$ satisfies

$$\int Dw \cdot Dv + awv - b(|u|^{p-2}u)_{x_i} v = 0.$$

Since this holds for every $v \in H^1$, we conclude that w is a weak solution of

$$-\Delta w + aw = b(|u|^{p-2}u)_{x_i}.$$

From Step 1 we deduce that the right-hand side belongs to L^q for every $q \in [2,\infty)$. Hence (12) implies that $w \in W^{2,q}$ for the same q. This in turn implies that $u \in W^{3,q}$ for the same range of q, and so $u \in W^{2,\infty}$, and moreover the second derivatives of u are $C^{0,\gamma}$ for every $\gamma < 1$.

(Note that by continuing in this fashion. we can easily show tat that $u \in W^{k,q}$ for every positive integer k and for all $q \in [u,\infty)$, so that in fact u is C^∞.)

3. Fix some multiindex α with $|\alpha| \le 2$. Since $D^\alpha u$ is $C^{0,1/2}$, one check that given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } \int_{B_1(x)} |D^\alpha u|^2 < \delta, \qquad \text{then } |D^\alpha u(x)| \le \varepsilon.$$

(Perhaps the easiest way to see this is to prove the contrapositive; see a similar argument in Step 6 below.) Then we can select R so large that

$$\int_{\mathbb{R}^n \setminus B_R(0)} |D^\alpha u|^2 < \delta.$$

It follows that $|D^\alpha u(x)| < \varepsilon$ for all $|x| \le R+1$. Thus $|D^\alpha u(x)| \to 0$ uniformly as $|x| \to \infty$.

This completes the proof of part **a** of the Proposition.

4. We now start the proof of **b**. For convenience we normalize (1) by assuming that $a = b = 1$; the general case follows by a small modification to our arguments,, or by rescaling.

For $\sigma > 0$, let $\eta_\sigma(x) := e^{\frac{|x|}{1+\sigma|x|}}$. This is a bounded, Lipschitz function. Since u is a solution of (11),

$$\int Du \cdot D(u\eta_\sigma) + \int u^2\eta_\sigma \;=\; \int |u|^p\eta_\sigma. \tag{15}$$

Elementary estimates, using the fact (easy to check) that $|D\eta_\sigma(x)| \le |\eta_\sigma(x)|$ for all x, show that

$$Du \cdot D(u\eta_\sigma) \ge \eta_\sigma|Du|^2 - |u|\,|Du|\;|D\eta_\sigma| \ge \frac{1}{2}\eta_\sigma|Du|^2 - \frac{1}{2}\eta_\sigma u^2$$

everywhere in $\mathbb{R}^n$. So (15) implies that

$$\int \eta_\sigma(|Du|^2 + u^2) \;\le\; 2\int |u|^p\eta_\sigma. \tag{16}$$

5. Now in view of Step 3, there exists $R > 0$ such that $|u|^{p-2} \le \frac{1}{4}$ for $|x| \ge R$. As a result,

$$\int |u|^p\eta_\sigma \;\le\; \int_{B_R(0)} |u|^p\eta_\sigma \;+\; \frac{1}{4}\int_{\mathbb{R}^n\setminus B_R(0)} |u|^2\eta_\sigma.$$

By combining this with (16) we find that

$$\int \eta_\sigma(|Du|^2 + u^2) \;\le 8\int_{B_R(0)} |u|^p\eta_\sigma \;\le\; 8\int_{B_R(0)} e^{|x|}\;|u|^p.$$

Now we let $\sigma \to 0$. The Monotone Convergence Theorem then yields

$$\int_{\mathbb{R}^n} e^{|x|}(|Du|^2 + u^2) \;\le 8\int_{B_R(0)} e^{|x|}\;|u|^p \;<\; \infty.$$

6. Let $f := |Du|^2+u^2$. It follows from part **a** of the Proposition that f is Lipschitz continuous. Let L denote the Lipschitz constant. Then for every $x, y \in \mathbb{R}^n$,

$$f(x) \ge \tfrac{1}{2}f(y) \quad\text{ if }\quad |x-y| \le f(y)/2L.$$

As a result, for every $y \in \mathbb{R}^n$,

$$\int e^{|x|}f(x) \;\ge\; \frac{f(y)}{2}\int_{\{x:|x-y|\le f(y)/2L\}} e^{|x|}\;dx.$$

And clearly $e^{|x|} \ge e^{|x|-\frac{f(y)}{2L}}$ in $\{x : |x-y| \le f(y)/2L\}$. Since f is bounded, this implies that

$$\int_{\{x:|x-y|\le f(y)/2L\}} e^{|x|}\;dx \;\ge\; c\left(\frac{f(y)}{2L}\right)^n e^{|x|-\frac{f(y)}{2L}} \;\ge\; c\left(\frac{f(y)}{2L}\right)^n e^{|x|}$$

By combining all these inequalities, we conclude that

$$f(y)^{n+1} \le Ce^{-|x|}$$

for some constant C (depending on u). This implies conclusion **b** (when a=b=1). □

Lemma 3. *Suppose that* $u \in H^1(\mathbb{R}^n)$ *solves*

$$-\Delta u + au - b|u|^{p-2}u = 0 \qquad \text{in } \mathbb{R}^n \tag{17}$$

for some $a > 0$ *and* $b \in \mathbb{R}$*. Then*

$$\int_{\mathbb{R}^n} |Du|^2\,dx + a\int_{\mathbb{R}^n} |u|^2\,dx \;=\; b\int_{\mathbb{R}^n} |u|^p\,dx \tag{18}$$

and

$$\frac{n-2}{2}\int_{\mathbb{R}^n} |Du|^2\,dx + a\frac{n}{2}\int_{\mathbb{R}^n} |u|^2\,dx \;=\; b\frac{n}{p}\int_{\mathbb{R}^n} |u|^p\,dx \tag{19}$$

Proof. **1**. Since u solves (1),

$$\int Du \cdot Dv + auv - b|u|^{p-2}uv \;=\; 0$$

for every $v \in H^1(\mathbb{R}^n)$. We deduce (18) by substituting u for v.

2. To prove (19), we multiply both sides of (17) by $x \cdot Du$ and integrate to obtain

$$-\int_{\mathbb{R}^n} \Delta u(x\cdot Du) + a\int_{\mathbb{R}^n} u(x\cdot Du) = b\int_{\mathbb{R}^n} |u|^{p-2}u(x\cdot Du).$$

We consider each term separately.

2a. One can check that

$$\begin{aligned}\Delta u(x\cdot Du) &= \sum_{i,j} u_{x_ix_i}u_{x_j}x_j \\ &= \sum_{i,j}(u_{x_i}u_{x_j}x_j)_{x_i} - \frac{1}{2}\sum_j(|Du|^2x_j)_{x_j} + \frac{n-2}{2}|Du|^2.\end{aligned}$$

(Simply expand the right-hand side.) The regularity and decay results proved above then imply that

$$\begin{aligned}\int_{\mathbb{R}^n} \Delta u(x\cdot Du) &= \lim_{R\to\infty}\int_{B_R} \Delta u(x\cdot Du) \\ &= \lim_{R\to\infty}\int_{B_R} \sum_{i,j}(u_{x_i}u_{x_j}x_j)_{x_i} - \frac{1}{2}\sum_j(|Du|^2x_j)_{x_j} + \frac{n-2}{2}|Du|^2 \\ &= \frac{n-2}{2}\int_{\mathbb{R}^n}|Du|^2 + \lim_{R\to\infty}\int_{\partial B_R}\sum_{i,j}(u_{x_i}u_{x_j}x_j)\nu_i - \frac{1}{2}\sum_j(|Du|^2x_j)\nu_j d\sigma \\ &= \frac{n-2}{2}\int_{\mathbb{R}^n}|Du|^2.\end{aligned}$$

2b. Similarly

$$u(x\cdot Du) = \sum_i ux_iu_{x_i} = \sum_i \frac{1}{2}x_i(u^2)_{x_i} = \sum_i \frac{1}{2}(x_iu^2)_{x_i} - \frac{n}{2}|u|^2$$

which implies that $\int_{\mathbb{R}^n} u(x\cdot Du) \;=\; -\int_{\mathbb{R}^n}\frac{n}{2}|u|^2$.

2c. And in exactly the same way,

$$|u|^{p-2}u(x\cdot Du) = \sum_i \frac{1}{p}x_i|u|^p_{x_i} = \sum_i \frac{1}{p}(x_i|u|^p)_{x_i} - \frac{n}{p}|u|^p,$$

so that $\int_{\mathbb{R}^n} |u|^{p-2}u(x\cdot Du) \;=\; -\int_{\mathbb{R}^n} \frac{n}{p}|u|^p$. By combining these facts, we arrive at (19). $\square$

Remark 1. *a slightly different way of carrying out the above computations starts from the fact that* (17) *is the Euler-Lagrange equation for the a functional involving a Lagrangian* $L(Du,u) = \frac{1}{2}|Du|^2 + \frac{a}{2}|u|^2 - \frac{b}{p}|u|^p$ *that does not depend on* x *(and hence exhibits "symmetry with respect to translations in the* x_j *variable, for* $j = 1,...,n$*." Thus we know on general ground that solutions of* (17) *have conservation laws associated to these symmetries. Concretely, in this case one can check that for any smooth function* u*,*

$$(-\Delta u + au - b|u|^{p-2}u)u_{x_i} \;=\; [\sum_j(-u_{x_i}u_{x_j})_{x_j}] + \frac{1}{2}(|Du|^2)_{x_i} + (\frac{|u|^2}{2} - \frac{|u|^p}{p})_{x_i}.$$

Thus the right-hand side equals zero for any smooth solution of (17)*. This can be thought of as a "conservation law", since the right-hand side is entirely in divergence form. One can obtain* (19) *by multiplying this conservation law*

$$[\sum_j(-u_{x_i}u_{x_j})_{x_j}] + \frac{1}{2}(|Du|^2)_{x_i} + (\frac{|u|^2}{2} - \frac{|u|^p}{p})_{x_i} \;= 0$$

by x_i*, summing from* $i = 1,\dots,n$ *and integrating by parts. On some level of course this is exactly the same argument.*

Note that Lemma 3 implies that

$$\int_{\mathbb{R}^n}|Du|^2\,dx \;=\; b(\frac{n}{2} - \frac{n}{p})\int_{\mathbb{R}^n}|u|^p\,dx \;=\; an\frac{p-2}{2p-np+2n}\int_{\mathbb{R}^n}|u|^2\,dx. \tag{20}$$

Next we state

Theorem 3 (symmetry and uniqueness)**.** *If* $u:\mathbb{R}^n \to \mathbb{R}$ *is a positive solution of* (1)*, then there exists some* $x_0 \in \mathbb{R}^n$ *such that* $x \mapsto u(x+x_0)$ *is radial.*

Moreover, there is a unique *positive, radial solution* $u_0 : \mathbb{R}^n \to (0,\infty)$ *of* (1)*.*

Thus every positive solution u *of* (1) *has the form* $u(x) = u_0(x - x_0)$*.*

The proof that every positive solution is radial uses a sophisticated form of the "method of moving planes". An easier example of the method of moving planes can be found in Evans, Section 9.5.2. A separate ingredient in the proof of Theorem 3 is a result establishing uniqueness for the radial problem.

We are interested in least-energy solutions of (1), and so we define

$$\mathcal{G} := \{u \in H^1(R^n)\;:\; u \text{ solves (1), and } I[u] \le I[v] \text{ if } v \in H^1(R^n) \text{ is any other solution of (1)}\}$$

A solution of (1) is sometimes called a *nonlinear bound state*, and a solution $u \in \mathcal{G}$ is called a *nonlinear ground state.*

Using the above one can prove

Theorem 4. *$\mathcal{G}$ is nonempty, and in fact*

$$u \in \mathcal{G} \iff u \text{ minimizes } I_D \text{ in } \mathcal{A}_{\Lambda_*} \tag{21}$$

$$\iff u(x) = \pm u_0(x - x_0) \quad \text{for some } x_0 \in \mathbb{R}^n \tag{22}$$

$$\implies u \text{ minimizes } G[\cdot]. \tag{23}$$

where $\mathcal{A}_{\Lambda_*}$ denotes the set defined in Theorem 1.

The proof of this theorem uses all the results described above, but no additional ingredients.