some theorems about determinants

Theorem 1 is a bit different from the presentation I gave during the lecture, and everything following Theorem 1 was not covered during the lecture. Please read it (and let me know if you find any misprints).

Theorem 1. *Suppose that u and v belong to $W^{1,n}(U;\mathbb{R}^n)$ and that $u-v\in W_0^{1,n}(U)$. Then*

$$\int_U \det Du\ dx \ = \ \int_U \det Dv\ dx.$$

(compare Theorems 1 and 2 in Section 8.1, Evans)

Proof. **1**. Given $u\in W^{1,n}(U)$, we define $j(u): U\to\mathbb{R}^n$ by

$$j^i(u) := \det\left(u_{x_1},\dots,u_{x_{i-1}},u,u_{x_{i+1}},\dots,u_{x_n}\right).$$

The notation on the right-hand side means: the determinant of the matrix whose columns are the given vectors, arranged in the given order. We claim that

$$\det Du = \frac{1}{n}\sum_{i=1}^n \partial_{x_i} j^i(u). \tag{1}$$

To see this, first note that the multilinearity of the determinant implies that

$$\begin{aligned}\partial_{x_i} j^i(u) &= \det\left(u_{x_1x_i},\dots,u_{x_{i-1}},u,u_{x_{i+1}},\dots,u_{x_n}\right)\\ &\qquad + \det\left(u_{x_1},u_{x_2x_i},\dots,u_{x_{i-1}},u,u_{x_{i+1}},\dots,u_{x_n}\right)\\ &\qquad + \ldots + \det\left(u_{x_1},\dots,u_{x_{i-1}},u,u_{x_{i+1}},\dots,u_{x_nx_i}\right)\\ &= \det Du + \sum_{k\neq i} T_{ik}\end{aligned}$$

where T_{ik} is the determinant of the matrix with $u_{x_kx_i}$ in the kth column, u in the ith column, and u_{x_ℓ} in the ℓth column if $\ell\notin\{i,k\}$. Basic properties of determinants imply that $T_{ik}=-T_{ki}$, so we find that

$$\sum_{i=1}^n \partial_{x_i} j^i(u) = \det Du + \frac{1}{n}\sum_{i=1}^n\sum_{k\neq i} T_{ik} = \det Du$$

since T_{ik} and T_{ki} cancel each other out.

2. Now suppose that u and v belong to $C^1(U;\mathbb{R}^n)$ and that $u-v$ is smooth with compact support in U. Then $j(u)=j(v)$ in a neighborhood of ∂U, and it follows that

$$\int_U \det Du = \int_U \nabla\cdot j(u) = \int_{\partial U} j(u)\cdot\nu = \int_{\partial U} j(v)\cdot\nu = \int_U \det Dv.$$

3. Now suppose that u belongs to $C^1(U;\mathbb{R}^n)$ and that $v=u+\phi$, for $\phi\in W_0^{1,n}(U)$. By definition of $W_0^{1,n}(U)$, there exists a sequence $\{\phi_k\}$ of smooth functions with compact support, such that $\|\phi_k-\phi\|_{W^{1,n}}\to 0$ as $k\to\infty$. Then we conclude that

$$\int_U \det Dv\ dx \ = \ \int_U \det D(u+\phi)\ dx = \lim_{k\to\infty}\int_U \det D(u+\phi_k)\ dx = \int_U \det Du$$

where the last equality holds for every k, by Step 2. Finally, a similar approximation argument allows us to conclude that

$$\int_U \det Du = \int_U \det Dv$$

if it is merely true that $u \in W^{1,n}(U)$ and $u - v \in W_0^{1,n}(U)$. $\square$

It should be obvious that the convergence $\|\phi_k - \phi\|_{W^{1,n}} \to 0$ as $k \to \infty$ implies that

$$\int_U \det D(u + \phi_k)\ dx \to \int_U \det D(u + \phi) \quad \text{ as } k \to \infty.$$

(This fact was used above.)

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For $1 \le k \le n$, let us write

$$I(k, n) := \{\alpha \in \mathbb{Z}^k \ : \ 1 \le \alpha_1 < \ldots < \alpha_k \le n\}.$$

For $\alpha, \beta \in I(k, n)$, we write $D^\alpha_\beta u$ to denote the $k \times k$ matrix with $u^{\alpha_i}_{x_{\beta_j}}$ in the (i, j) position. We will also write u^α to denote the (column) vector

$$u^\alpha = \begin{pmatrix} u^{\alpha_1} \\ \vdots \\ u^{\alpha_k} \end{pmatrix}$$

Then $D^\alpha_\beta u = (u^\alpha_{x_{\beta_1}}, \ldots, u^\alpha_{x_{\beta_k}})$ where the right-hand side denotes the matrix whose columns are the given vectors arranged in the given order.

Having introduced this notation, we can see that

$$\det D^\alpha_\beta u = \frac{1}{k} \sum_{i=1^k} \partial_{x_{\beta_i}} j^i_\beta(u^\alpha) \tag{2}$$

where

$$j^i_\beta(u^\alpha) = \det(u^\alpha_{x_{\beta_1}}, \ldots, u^\alpha_{x_{\beta_{i-1}}}, u^\alpha, u^\alpha_{x_{\beta_{i+1}}} u^\alpha_{x_{\beta_k}}). \tag{3}$$

This follows by exactly same the calculation carried out in Step 1 of the above proof.

A determinant of a sub-matrix of Du is often called a "minor of Du". We may also speak of a "minor of order k" or a "$k \times k$ minor" if the submatrix in question is a $k \times k$ matrix. An easy adaptation of the proof given above shows that $L(Du) := \det D^\alpha_\beta u$ is a null Lagrangian for every α, β as above.

Next we prove

Lemma 1. *(Weak continuity of determinants) Assume that $n < q < \infty$ and that*

$$u_\ell \rightharpoonup u \text{ weakly in } W^{1,q}(U; \mathbb{R}^n).$$

Then

$$\det Du_\ell \rightharpoonup \det Du \quad \text{ weakly in } L^{q/n}(U).$$

(Compare the Lemma in Section 8.2.4b, Evans; this is the same lemma with a slightly different proof.)

For the proof we will need the following standard (and very useful) fact.

Lemma 2. *Suppose that $\{f_\ell\}, \{g_\ell\}$ are sequences of functions and f, g are functions such that*

$$f_\ell \to f \text{ strongly in } L^p(U), \qquad g_\ell \to g \text{ weakly in } L^q(U)$$

for some $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$. Then

$$f_\ell g_\ell \rightharpoonup fg \text{ weakly in } L^r(U).$$

Proof of Lemma 2. Fix $h \in L^{r'}$, where $\frac{1}{r} + \frac{1}{r'} = 1$. We must show that

$$\int f_\ell g_\ell h \, dx \quad \to \int fgh \, dx$$

as $k \to \infty$. To do this, we write

$$\int (f_\ell g_\ell h - fgh) \, dx \ = \ \int (f_\ell - f) g_\ell h \, dx + \int (g_\ell - g) fh \, dx.$$

For the first integral, since $\frac{1}{p} + \frac{1}{q} + \frac{1}{r'} = 1$, Holder's inequality implies that

$$|\int (f_\ell - f) g_\ell h \, dx| \le \|f_\ell - f\|_p \|g_\ell\|_q \|h\|_{r'} \le C \|f_\ell - f\|_p \to 0$$

as $k \to \infty$. that a weakly convergent sequence is bounded, which implies in particular that there exists some C such that $\|g_\ell\|_q \le C$. This fact is a consequence of the Banach-Steinhaus Theorem (also known as the Uniform Boundedness Principle).

Also, let q' be the Holder dual of q, so that $\frac{1}{q'} = 1 - \frac{1}{q}$. Then $\frac{1}{q'} = \frac{1}{p} + \frac{1}{r'}$, so Holder's inequality implies that $\|fh\|_{q'} \le \|f\|_p \|h\|_{r'}$. Thus $fg \in L^{q'}$, and the weak convergence $g_\ell \to g$ weakly in L^q implies that $\int (g_\ell - g) fh \, dx \quad \to 0$ as $k \to \infty$. □

Using the Lemma we give the

Proof of Lemma 1. We will in fact prove that if $\alpha, \beta \in I(k, n)$ for any $k \in \{1, \ldots, n\}$, and

$$\text{if } u_\ell \to u \text{ weakly in } W^{1,q}(U), \text{ then } \det D^\alpha_\beta u_\ell \rightharpoonup \det D^\alpha_\beta u \text{ weakly in } L^{q/k} \tag{4}$$

We prove this by induction on k.

The case $k = 1$ is clear.

Suppose we have proved (4) for $1, 2, \ldots, k-1$, for $k \le n$. Fix $\alpha, \beta \in I(k, n)$, and assume that $u_\ell \to u$ weakly in $W^{1,q}(U)$.

For any smooth function v with compact support, note from (2), (3) that

$$\int_U v \det D^\alpha_\beta u_\ell \ dx \ = \ - \int_U \sum_i v_{x_i} \, j^i_\beta(u^\alpha_\ell) \ dx$$

after integration by parts. Note that for each i and ℓ,

$$j^i_\beta(u^\alpha_\ell) = \text{ sum of terms of the form } (u^\ell_{\alpha_k}) \times [\text{ minor of } Du_\ell \text{ of order (k-1)}].$$

Rellich's compactness theorem implies that that $u_\ell^{\alpha_k} \to u^{\alpha_k}$ strongly in L^q as $q \to \infty$, and the induction hypothesis implies that every sequence of minors of Du_ℓ of order $k-1$ converges weakly in $L^{q/(k-1)}$ to the corresponding minor of Du. Thus it follows from Lemma 2 that $j^i_\beta(u^\alpha_\ell) \rightharpoonup j^i_\beta(u^\alpha)$ as $\ell \to \infty$. Consequently,

$$\int_U v \det D^\alpha_\beta u_\ell \; dx \to -\int_U \sum_i v_{x_i} \, j^i_\beta(u^\alpha) \; dx = \int_U v \det D^\alpha_\beta u \; dx \tag{5}$$

as $\ell \to \infty$.

Note that $|\det D^\alpha_\beta u_\ell| \le C |Du_\ell|^k$ pointwise, which implies that $\|\det D^\alpha_\beta u_\ell\|_{q/k} \le C \| \, |Du_\ell|^k \, \|_{q/k} \le \|Du_\ell\|_q^k \le C$ (again by the Banach-Steinhaus Theorem.) Thus every subsequence of $\{\det D^\alpha_\beta u_\ell \}$ has a subsequence that converges weakly in $L^{q/k}$. However, (5) implies that the only possible weak limit of any convergent subsequence is $\det D^\alpha_\beta u$. It follows by standard arguments (which you should know!) that the whole sequence converges weakly in $L^{q/k}$, and that the weak limit is $\det D^\alpha_\beta u$.

$\square$