PDEII, winter 2008,

Summary of existence and uniqueness results for 2nd order linear elliptic PDE

We are interested in studying existence and uniqueness of solutions of the problem

$$
Lu = f \quad \text{in } U, \qquad\qquad u = 0 \quad \text{on } \partial U \tag{1}
$$

where U is a bounded open subset of $\mathbb{R}^n$ with C^1 boundary and

$$
Lu = -\sum_{i,j} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_i b^i(x)u_{x_i} + c(x)u.
$$

The coefficients are all assumed to be bounded and measurable on U, and we make the crucial ellipticity assumption

$$
\sum_{i,j} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \text{for a.e. } x \in U \text{and every } \xi \in \mathbb{R}^n. \tag{2}
$$

We will generally assume that $f \in H^{-1}(U)$. This of course allows us to consider $f \in L^2(U)$, since $L^2(U) \subset H^{-1}(U)$.

Recall that we say that u is a weak solution of (1) if $u \in H^1_0(U)$ and

$$
\int_U \left[\sum_{i,j} a^{ij}(x)u_{x_i}v_{x_j} + \sum_i b^i(x)u_{x_i}v + c(x)uv \right] dx = \langle f, v\rangle \qquad \text{for all } v \in H^1_0(U). \tag{3}
$$

We will write $B[u, v]$ to indicate the expression on the left-hand side of (3). Note that if u is smooth enough, then $B[u, v] = (Lu, v)_{L^2}$.

Here is a summary of the overall strategy we have used.

(1) Change the problem to an easier problem! Prove existence and uniqueness for weak solutions of the problem

$$
L_\gamma u := Lu + \gamma u = f \quad \text{in } U, \qquad\qquad u = 0 \quad \text{on } \partial U \tag{4}
$$

* main tool: the Lax-Milgram Theorem, which is based on the Riesz Representation Theorem, which asserts that given a (real) Hilbert space H and a bounded linear functional $\ell : H \to \mathbb{R}$, there exists a unique $u \in H$ such that $(u, v) = \ell(v)$ for all $v \in H$.

The bilinear form $B[\cdot, \cdot]$ appearing in the Lax-Migram Theorem can be thought of as resembling a Hilbert space inner product without the symmetry assumption. (In fact if B is symmetric, ie if $B[u, v] = B[v, u]$ for all u and v, then B is equivalent to the Hilbert space inner product.)

* The ellipticity assumption and the extra term γu in (4) are needed in verifying the hypothesis $B_\gamma[u, u] \geq \|u\|^2_{H^1_0}$, which is needed to apply the abstract Lax-Milgram Theorem to the concrete problem (4). Here $B_\gamma[u, v] := B[u, v] + \gamma \int_U uv\, dx$.

* Note that in *this* discussion, $(\cdot, \cdot)$ denotes an abstract Hilbert space inner product for purposes of the abstract Lax-Milgram Theorem, and the inner product in the Hilbert space H^1_0 in the proof of the existence and uniqueness theorem for (4).

(2) Once we have an existence theorem at hand for (4), we can rephrase all questions about (1) as questions about an equation involving a compact operator.

(a) We write $u = L_\gamma^{-1} f$ if u is the unique weak solution of (4).

(b) Then

(5) $$u \text{ is a weak solution of (1)} \iff (I-K)u := g \text{ for } K := \gamma L_\gamma^{-1}, \quad g = L_\gamma^{-1} f.$$

Moreover, it is easy to check that $K : L^2 \to L^2$ is compact. Thus the abstract Fredholm Alternative for equations of the form $(I-K)u = g$ implies a similar Fredholm alternative for (1). It follows that (1) *has a weak solution for all* $f \in H^{-1}(U)$ *if and only if the homogeneous problem (ie* $f = 0$*) has a unique weak solution* (among other conclusions).

(c) Similarly, $Lu = \lambda u$ in U, $u = 0$ on ∂U, if and only if $L_\gamma u = (\lambda+\gamma)u$, or equivalently $Ku = \frac{\gamma}{\gamma+\lambda}u$.
In other words,

(6) $$\lambda \text{ is an eigenvalue for } u \iff \frac{\gamma}{\gamma+\lambda} \text{ is an eigenvalue of } K$$

Thus abstract results about eigenvalues of compact operators (there are at most countably many eigenvalues, all of finite multiplicity, and the only possible accumulation point of the sequence of eigenvalues is 0) translate to results about eigenvalues of L. This yields: *L has at most countably many eigenvalues, all of finite multiplicity, and if there are countably many then they form a sequence tending to $+\infty$.*
(Note that if $\lambda + \gamma \leq 0$, ie if $-\lambda > \gamma$, then the verification that B_γ satisfies the hypotheses of the Lax-Milgram theorem implies uniqueness of solutions of the problem $Lu - \lambda u = 0$ in U, with boundary conditions $u = 0$ on ∂U. Thus every eigenvalue of L is greater than $-\gamma$.)

(d) By combining the two previous results, we find that *the problem*

$$Lu = \lambda u + f \ \textit{ in } U, \quad u = 0 \ \textit{ on } \partial U$$

has a unique weak solution for every $f \in H^{-1}(U)$ unless λ is an eigenvalue of L, which implies that the problem is uniquely solvable for all but countably many values of λ.

(e) Furthermore, if $b^i = 0$ for all i, then $K = \gamma L_\gamma^{-1}$ is symmetric (that is, $(Ku, v) = (u, Kv)$ for all $u, v \in L^2(U)$) and so one can appeal to abstract theory about the eigenvalues and eigenfunctions of compact*symmetric* operators, and deduce conclusions about eigenvalues and eigenfunctions of L. This leads to: *if $b^i = 0$ for all i, then $L^2(U)$ has an orthonormal basis consisting of eigenfunctions of L.*

(f) In all of the above theory involving the compact operator K, the Hilbert space is $L^2(U)$, and $(\cdot,\cdot)$ denotes the L^2 inner product, rather than the H^1_0 inner product as earlier. THis is because it is natural to prove that K is compact *as an operator on L^2*.

(g) In all of this theory, the crucial ellipticity assumption is used in two ways. First, it guarantees that the Lax-Milgram machinery is applicable, so that we can appeal to earlier results to rewrite our original problem (1) as in (5) and obtain further equivalences such as (6).
Second and more subtly, the ellipticity is resposible for the estimate $\|Ku\|_{H^1_0} \leq C\|u\|_{L^2}$, and this in turn (together with Rellich's compactness theorem) is responsible for the fact that $K : L^2 \to L^2$ is compact, and hence that all the machinery used above is applicable.