MAT 235 TEST #3

MODEL SOLUTIONS

MARCH 5, 2002

1. a) $z_x = 4x/z$ and $z_y = 4y/z$

Then $\sqrt{1 + z_x^2 + z_y^2} = \sqrt{5}$

Using polar coordinates:

$$A = \int_0^{2\pi} \int_1^2 \sqrt{5}\, r\, dr\, d\theta = 3\pi\sqrt{5}$$

b) $-1 \le x \le 1, \quad 0 \le y \le \sqrt{1-x^2}, \quad 0 \le z \le \sqrt{1-x^2-y^2}$

Using spherical coordinates:

$0 \le \theta \le \pi, \quad 0 \le \phi \le \frac{\pi}{2}, \quad 0 \le \rho \le 1$

$$\int_0^{\pi} \int_0^{\pi/2} \int_0^1 \rho^6 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \Big[\theta\Big]_0^{\pi} \Big[-\cos\phi\Big]_0^{\pi/2} \Big[\tfrac{1}{7}\rho^7\Big]_0^1$$

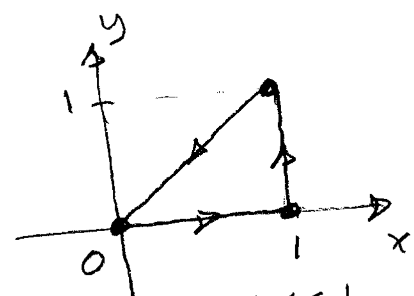
$$= \frac{\pi}{7}$$

(2) a) $\int_C z\,ds = \int_0^1 \frac{2}{3} t^3 \sqrt{4 + t^4 + 4t^4}\,dt$

$$= \int_0^1 \frac{2}{3} t^3 \sqrt{4+5t^4}\,dt$$

$$= \frac{1}{45}\left(4+5t^4\right)^{3/2}\Big]_0^1$$

$$= \frac{1}{45}(27-8) = \frac{19}{45}$$



$$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases}$$

b) $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 2y$

$$\int_0^1 \int_0^x (1-2y)\,dy\,dx$$

$$= \int_0^1 \left[y - y^2\right]_0^x dx$$

$$= \int_0^1 (x - x^2)\,dx$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

③ a) $\operatorname{div} F = 1$

$\operatorname{curl} F = (a-1, 0, 0)$

$F \times \operatorname{curl} F = (0, a(a-1)y, (1-a)z)$

$\operatorname{div}(F \times \operatorname{curl} F) = a(a-1) + 1 - a$

$= a^2 - 2a + 1$

For $a^2 - 2a + 1 = 1$ $\quad a = 0$ or $a = 2$

b) Take $g(x, y) = e^{2y} - e^{-2y}\cos x$

Then $\int_C G \cdot dr = g(\pi, -1) - g(0, 1)$

$= e^{-2} + e^{2} - e^{2} + e^{-2}$

$= 2e^{-2}$

(4)

$$M = \int_0^1 \int_0^{2-y} \int_0^{\sqrt{1-y^2}} \frac{2z}{1+y}\, dz\, dx\, dy$$

$$= \int_0^1 \int_0^{2-y} \frac{1-y^2}{1+y}\, dx\, dy$$

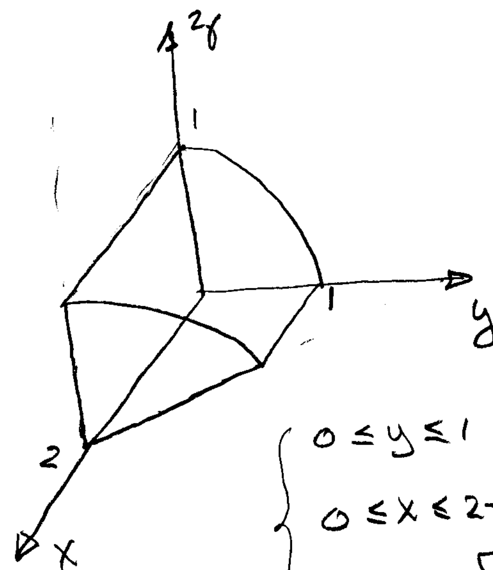
$$= \int_0^1 (1-y)(2-y)\, dy$$

$$= \int_0^1 (2-3y+y^2)\, dy$$

$$= 2 - \frac{3}{2} + \frac{1}{3}$$

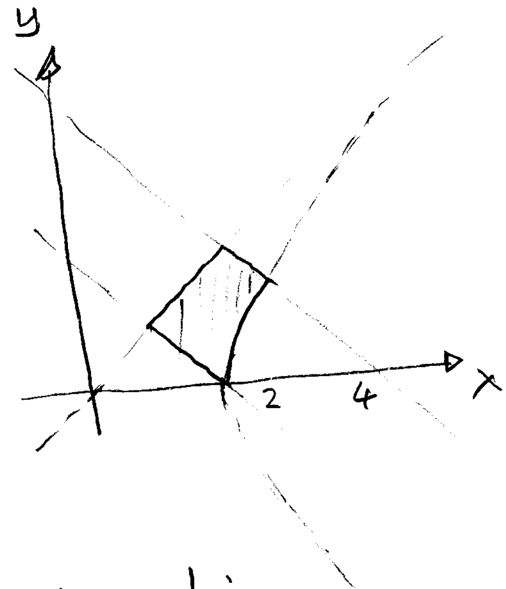
$$= \frac{12-9+2}{6}$$

$$= \frac{5}{6}$$



$$\begin{cases} 0 \le y \le 1 \\ 0 \le x \le 2-y \\ 0 \le z \le \sqrt{1-y^2} \end{cases}$$

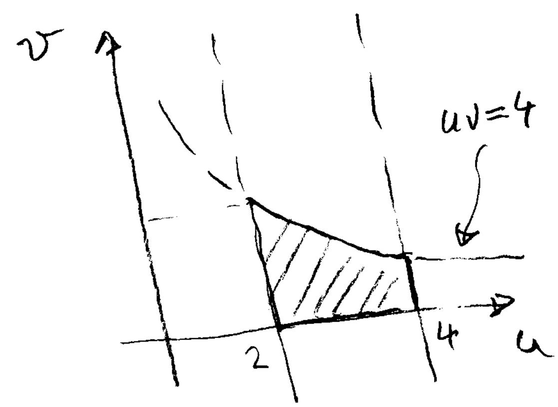
⑤ Make $\begin{cases} u = x+y \\ v = x-y \end{cases}$



Then $\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$

And $\left| \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \right| = \frac{1}{2}$

The corresponding region of integration in the uv-system:



$\begin{cases} 2 \le u \le 4 \\ 0 \le v \le \frac{4}{u} \end{cases}$

$\int_2^4 \int_0^{\frac{4}{u}} \frac{1}{2} u^2 \, dv \, du$

$= \int_2^4 2u \, du$

$= u^2 \Big]_2^4$

$= 12$