University of Toronto
Department of Mathematics

MAT 235Y1Y
Calculus II

TERM TEST # 3
Tuesday, March 13, 2001
Time allowed: 2 hours

Last Name: MODEL SOLUTIONS

Given Name: ____________________

Student Number: ____________________

Lecture Section: __________ **Prof. Name:** __________

INSTRUCTIONS:

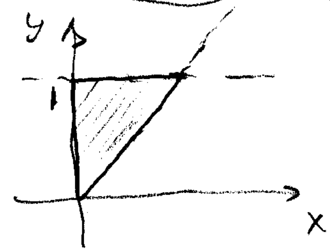
- No aids are allowed.

FOR TA USE ONLY	
Question	Mark
1	/ 15
2	/ 15
3	/ 15
4	/ 20
5	/ 20
6	/ 15
TOTAL	

1. a) Convert the integral given below in polar coordinates to an equivalent integral in Cartesian coordinates and evaluate the latter.

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\csc\theta} r^4 \sin^2\theta \cos\theta \, dr \, d\theta.$$

Region of integration: $\begin{cases} \frac{\pi}{4} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le \csc\theta \end{cases}$



$\begin{cases} 0 \le y \le 1 \\ 0 \le x \le y \end{cases}$

2 marks

(7) Then $I = \int_0^1 \int_0^y x y^2 \, dx \, dy$

3 marks

$= \int_0^1 \frac{1}{2} y^4 = \boxed{\frac{1}{10}}$

2 marks

b) Find the volume of the region in the first octant that lies between the cylinders $r = 1$ and $r = 2$ and between the xy-plane and the surface $z = xy$.

Use "cyl. coord". Our solid is: $\begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 1 \le r \le 2 \\ 0 \le z \le r^2 \sin\theta \cos\theta \end{cases}$

(8) $V = \int_0^{\pi/2} \int_1^2 \int_0^{r^2 \sin\theta\cos\theta} r \, dz \, dr \, d\theta$

2 marks

3 marks

$= \int_0^{\pi/2} \int_1^2 r^3 \sin\theta\cos\theta \, dr \, d\theta = \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} \left[\frac{1}{4} r^4\right]_1^2$

$= \left(\frac{1}{2}\right)\left(\frac{15}{4}\right) = \boxed{\frac{15}{8}}$

3 marks

2. Find the mass of the solid ~~enclosed by the~~ under the sphere $x^2+y^2+z^2=1$ and above the cone $z=\sqrt{x^2+y^2}$. If the density is $\delta(x,y,z)=\sqrt{x^2+y^2+z^2}$.

(15)

Use spherical coord:

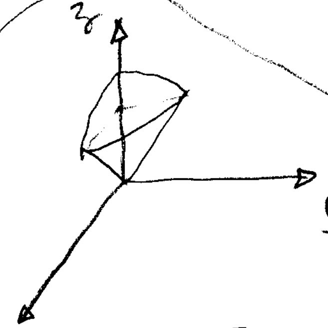
5 marks

$$M=\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$=(2\pi)\left(-\cos\phi\right)_0^{\pi/4}\left(\tfrac{1}{4}\rho^4\right)_0^1$$

$$=\left(\frac{\pi}{2}\right)\left(1-\frac{\sqrt{2}}{2}\right)=\boxed{\frac{\pi}{4}\left(2-\sqrt{2}\right)}$$

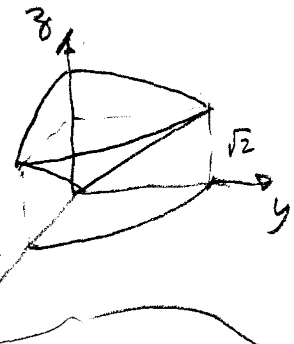
5 marks



$$\begin{cases} 0\le\theta\le 2\pi \\ 0\le\phi\le\frac{\pi}{4} \\ 0\le\rho\le 1 \end{cases}$$

5 marks

3. Convert $\int_0^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} y^2\, dz\, dx\, dy$



a) to cylindrical coordinates

Region of integration: $\begin{cases} 0 \le y \le 2 \\ -\sqrt{2-y^2} \le x \le \sqrt{2-y^2} \\ \sqrt{x^2+y^2} \le z \le \sqrt{4-x^2-y^2} \end{cases}$

(5) $\boxed{\int_0^{\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r^3 \sin^2\theta \, dz\, dr\, d\theta}$ ← 3 marks

2 marks → $\begin{cases} 0 \le \theta \le \pi \\ 0 \le r \le \sqrt{2} \\ r \le z \le \sqrt{4-r^2} \end{cases}$

b) to Spherical coordinates.

(5) $\boxed{\int_0^{\pi} \int_0^{2} \int_0^{\pi/4} \rho^4 \sin^2\theta \sin^3\phi \, d\phi\, d\rho\, d\theta}$

3 marks

$\begin{cases} 0 \le \theta \le \pi \\ 0 \le \rho \le 2 \\ 0 \le \phi \le \frac{\pi}{4} \end{cases}$

2 marks

c) Evaluate one of the integrals.

$$\int_0^{\pi} \int_0^{2} \int_0^{\pi/4} \rho^4 \sin^2\theta \sin^3\phi \, d\phi\, d\rho\, d\theta = \int_0^{\pi} \sin^2\theta\, d\theta \int_0^2 \rho^4 d\rho \int_0^{\frac{\pi}{4}} \sin^3\phi\, d\phi$$

(5)

$$= \left(\frac{32}{5}\right) \int_0^{\pi} \frac{1}{2}(1-\cos(2\theta))\, d\theta \int_0^{\pi/4} (\sin\phi - \sin\phi\cos^2\phi)\, d\phi$$

$$= \left(\frac{32}{5}\right)\left(\frac{\pi}{2}\right)\left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi/4} = \frac{16\pi}{5}\left[-\frac{\sqrt{2}}{2} + \frac{2\sqrt{2}}{24} + 1 - \frac{1}{3}\right]$$

1 marks

$$= \boxed{\frac{4\pi}{15}\left[8 - 5\sqrt{2}\right]}$$

2 marks

2 marks

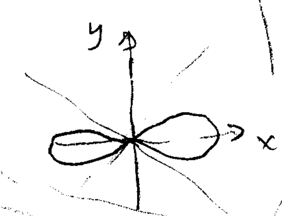
4. Find the valume of the region bounded by $z = x^2 + y^2, z = 0,$ and $(x^2 + y^2)^2 = x^2 - y^2$.

(20)

using "cylind. coord."

$z = r^2 \quad z = 0 \quad r^4 = r^2\cos(2\theta)$

$r^2 = \cos(2\theta)$



$$V = 4\int_0^{\pi/4}\int_0^{\sqrt{\cos(2\theta)}}\int_0^{r^2} r\,dz\,dr\,d\theta$$

7 marks

$$= 4\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos(2\theta)}} r^3\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \cos^2(2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos(4\theta))\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_0^{\pi/4}$$

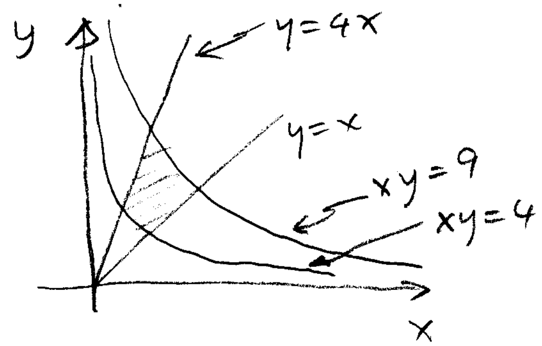
$$= \frac{1}{2}\left[\frac{\pi}{4} + 0\right] = \boxed{\frac{\pi}{8}}$$

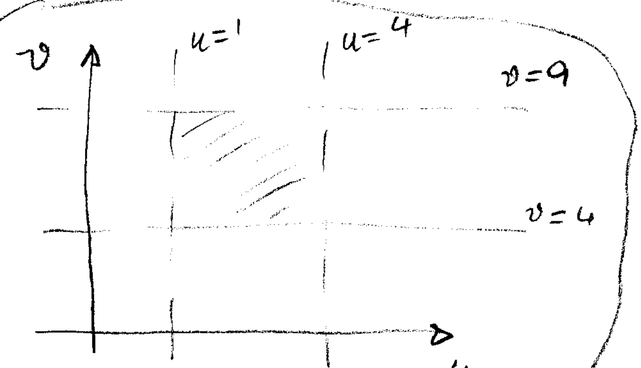
6 marks

7 marks

5. Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 4$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation $u = \frac{y}{x}$, $v = xy$ to rewrite $\int_R\int (\sqrt{y/x} + \sqrt{xy})\, dx\, dy$ as an integral over an appropriate region R^* in the uv-plane. Then evaluate the uv-integral over R^*.

(20)





$$\begin{cases} 1 \le u \le 4 \\ 4 \le v \le 9 \end{cases}$$

5 marks

$$x = \sqrt{\frac{v}{u}} \qquad y = \sqrt{uv}$$

$$\frac{dx}{du} = -\frac{1}{2}u^{-3/2}v^{1/2} \qquad \frac{dx}{dv} = \frac{1}{2}u^{-1/2}v^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2}v^{1/2} \qquad \frac{dy}{dv} = \frac{1}{2}u^{1/2}v^{-1/2}$$

$$J = \left| -\frac{1}{4}u^{-1} - \frac{1}{4}u^{-1} \right| = \frac{1}{2}u^{-1}$$ ← 5 marks

5 marks →
$$\iint_R \left[\sqrt{y/x} + \sqrt{xy}\right] dA = \int_1^4 \int_4^9 (\sqrt{u} + \sqrt{v})\left(\frac{1}{2}u^{-1}\right) dv\, du$$

$$= \frac{1}{2}\int_1^4 \left[5u^{-1/2} + \frac{2}{3}(27-8)u^{-1}\right] du$$

$$= \frac{1}{2}\left[(10)(1) + \frac{38}{3}\ln 4\right] = \boxed{5 + \frac{38}{3}\ln 2}$$ ← 5 marks

6. Find the area of the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 4$.

(15)

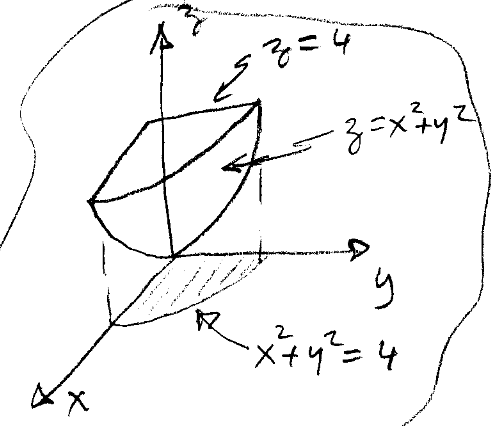
$$A = \int_0^{2\pi}\int_0^{2} \sqrt{1+4r^2}\, dr\, d\theta$$

5 marks

$$= (2\pi)\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_0^2$$

$$= \boxed{\frac{\pi}{6}\left[17^{3/2} - 1\right]}$$

5 marks



5 marks