UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS

MAT 235 Y - CALCULUS FOR PHYSICAL AND LIFE SCIENCES
FALL-WINTER 1995-96

TEST #3 - MARCH 19, 1996

NAME: MODEL SOLUTIONS STUDENT No: ____________
(Family name. Please PRINT.) (Given name.)

INSTRUCTIONS: This test consists of six questions. The value of each question is indicated (in brackets) by the question number. Total marks: 45.
Show all your work in all questions. Give your answers in the space provided. Use both sides of the paper, if necessary. Do not tear out any pages.
No calculators or any other aids are permitted. This test is worth 15% of your course grade. Keep your student card visible on your table. Time allowed: 2 hours.

1. (7 marks) Compute the surface area of the part of the graph of

$z = x^2 + xy - y^2$ lying over the region $x^2 + y^2 \le 3$, $0 \le y \le x$.

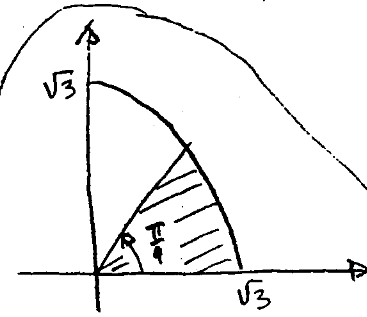
Soln: $z_x = 2x+y \qquad z_y = x-2y$

$z_x^2 = 4x^2+4xy+y^2 \qquad z_y = x^2-4xy+4y^2$

$z_x^2 + z_y^2 = 5x^2+5y^2$

$A = \iint_T \sqrt{1+5(x^2+y^2)}\, dA$

2 marks



T: $0 \le r \le \sqrt{3}$
$0 \le \theta \le \frac{\pi}{4}$

3 marks

$= \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{3}} r\sqrt{1+5r^2}\, dr\, d\theta$

$= \Big[\theta\Big]_0^{\pi/4} \Big[\frac{1}{15}(1+5r^2)^{3/2}\Big]_0^{\sqrt{3}}$ ← 2 marks

$= \left(\frac{\pi}{4}\right)\left(\frac{1}{15}\right)(64-1) = \boxed{\frac{21\pi}{20}}$

2. (8 marks) Given the triple integral $\iiint_R \frac{1}{2+x}\,dV$, where R is the region $0 \le x \le y \le 2$, $0 \le z \le 4 - x^2$.

a) Express the triple integral as an iterated integral in which the inner integral is with respect to z and the outer integral is with respect to y.

b) Evaluate the integral.

Soln:

a) $$\iiint_R \frac{1}{2+x}\,dV = \int_0^2 \int_0^y \int_0^{4-x^2} \frac{1}{2+x}\,dz\,dx\,dy$$

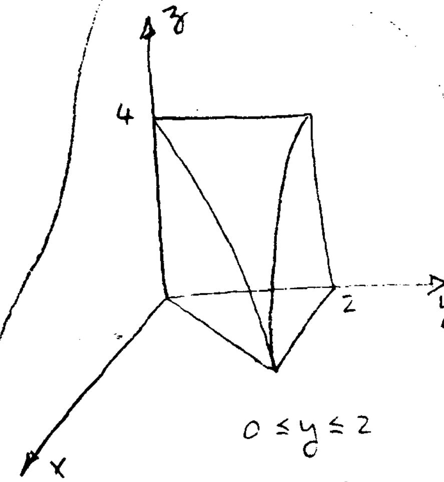
2 marks

b) $$\iiint_R \frac{1}{2+x}\,dV = \int_0^2 \int_0^y \frac{z}{2+x}\Big]_0^{4-x^2} dx\,dy$$

$$= \int_0^2 \int_0^y (2-x)\,dx\,dy = \int_0^2 2x - \frac{x^2}{2}\Big]_0^y dy$$

$$= \int_0^2 \left(2y - \frac{y^2}{2}\right) dy = y^2 - \frac{y^3}{6}\Big]_0^2 = 4 - \frac{8}{6} = \boxed{\frac{8}{3}}$$

4 marks



$0 \le y \le 2$

$0 \le x \le y$

$0 \le z \le 4 - x^2$

2 marks

3. (8 marks) Determine the coordinates of the centroid of the region $x^2 + y^2 + z^2 \le 4$, $0 \le z \le \sqrt{3x^2 + 3y^2}$. Hint: spherical coordinates.

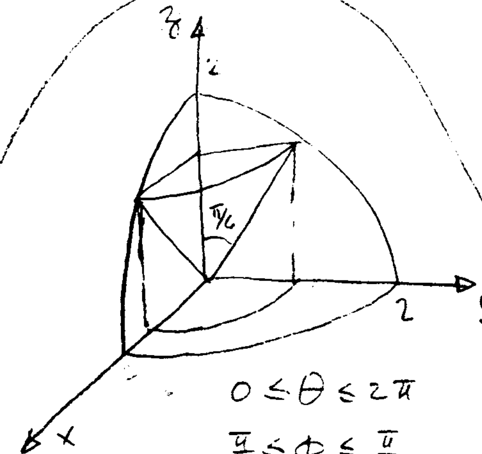
Soln.

2 marks

$$M = \int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \Big[\theta\Big]_0^{2\pi} \Big[-\cos\phi\Big]_{\pi/6}^{\pi/2} \left[\frac{\rho^3}{3}\right]_0^2$$

$$= (2\pi)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{8}{3}\right) = \frac{8\pi\sqrt{3}}{3}$$



$0 \le \theta \le 2\pi$

$\frac{\pi}{6} \le \phi \le \frac{\pi}{2}$

$0 \le \rho \le 2$

3 marks

$$M_{xy} = \int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^2 \rho^3 \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta$$

$$= \Big[\theta\Big]_0^{2\pi} \left[\frac{1}{2}\sin^2\phi\right]_{\pi/6}^{\pi/2} \left[\frac{\rho^4}{4}\right]_0^2 = (2\pi)\left(\frac{1}{2}\right)\left(1 - \frac{1}{4}\right)(4) = 3\pi$$

2 marks

$$\bar{z} = \frac{M_{xy}}{M} = \frac{3\pi}{\frac{8\pi\sqrt{3}}{3}} = \boxed{\frac{3\sqrt{3}}{8}}$$

$\boxed{\bar{x} = \bar{y} = 0 \text{ because of the symmetry}}$ ← 1 mark

4. (8 marks) Evaluate $\int_C F\cdot dr$, where $F(x,y,z) = \left(z+z^2, \frac{1}{1+z^2}, y-x\right)$, and C is the curve consisting of the arc $x^2 + z^2 = 4$, $y = 0$, $z \ge 0$, from (0, 0, 2) to (2, 0, 0), followed by the line segment from (2, 0, 0) to (0, 3, 1).

Soln. Along C_1:

3 marks

$$F = \left(2\cos\theta + 4\cos^2\theta, \frac{1}{1+2\cos^2\theta}, -2\sin\theta\right)$$

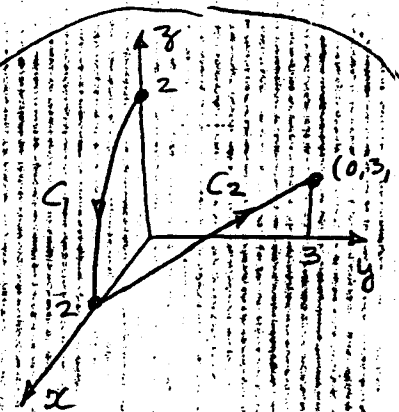
$$dr = (2\cos\theta, 0, -2\sin\theta)\, d\theta$$

$$\int_{C_1} F\cdot dr = \int_0^{\pi/2} (4\cos^2\theta + 8\cos^3\theta + 4\sin^2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (4 + 8\cos\theta - 8\sin^2\theta\cos\theta)\, d\theta$$

$$= 4\theta + 8\sin\theta - \frac{8}{3}\sin^3\theta \Big]_0^{\pi/2}$$

$$= 2\pi + 8 - \frac{8}{3} = 2\pi + \frac{16}{3}$$



$$C_1: \begin{cases} x = 2\sin\theta \\ y = 0 \\ z = 2\cos\theta \end{cases} \quad 0 \le \theta \le \frac{\pi}{2}$$

$$C_2: \begin{cases} x = 2-2t \\ y = 3t \\ z = t \end{cases} \quad 0 \le t \le 1$$

2 marks

Along C_2:

3 marks

$$F = \left(t+t^2, \frac{1}{1+t^2}, 5t-2\right)$$

$$dr = (-2, 3, 1)$$

$$\int F\cdot dr = \int_0^1 \left(-2t - 2t^2 + \frac{3}{1+t^2} + 5t - 2\right) dt = \int_0^1 \left(3t - 2 - 2t^2 + \frac{3}{1+t^2}\right) dt$$

$$\frac{3t^2}{2} - 2t - \frac{2t^3}{3} + 3\arctan t \Big]_0^1 = \frac{3}{2} - 2 - \frac{2}{3} + \frac{3\pi}{4} = \frac{9-12-4}{6} = \frac{3\pi}{4} - \frac{7}{6}$$

$$\int F\cdot dr = 2\pi + \frac{16}{3} + \frac{3\pi}{4} - \frac{7}{6} = \boxed{\frac{11\pi}{4} + \frac{25}{6}}$$

5. (7 marks) For each of the following two integrals, determine whether the line integral is path independent. Evaluate the integral only if it is path independent.

a) $\displaystyle\int_{(2,-1,0)}^{(0,1,\pi)} F\cdot dr$, where $F(x,y,z)=(y^2+\sin z, 2xy+\cos z, x\cos z)$.

b) $\displaystyle\int_{(-1,2)}^{(3,0)} F\cdot dr$, where $F(x,y)=\left(\dfrac{2xy}{1+y^2}, \dfrac{x^2-2y-x^2y^2}{(1+y^2)^2}\right)$.

3 marks

a) Let $F(x,y,z)=(u,v,w)$.

$\dfrac{\partial u}{\partial y}=2y=\dfrac{\partial v}{\partial x}$, $\dfrac{\partial u}{\partial z}=\cos z=\dfrac{\partial w}{\partial x}$, but

$\dfrac{\partial v}{\partial z}=-\sin z$ while $\dfrac{\partial w}{\partial y}=0$, that is $\dfrac{\partial v}{\partial z}\neq\dfrac{\partial w}{\partial y}$

The line integral is not path independent

1 mark

b) Let $F(x,y)=(u,v)$

$\dfrac{\partial u}{\partial y}=\dfrac{2x(1+y^2)-2y(2xy)}{(1+y^2)^2}=\dfrac{2x-2xy^2}{(1+y^2)^2}=\dfrac{\partial v}{\partial y}$ (Path. indep.)

2 marks

Finding a potential function $f(x,y)$:

$f(x,y)=\displaystyle\int\frac{2xy}{1+y^2}\,dx=\frac{x^2y}{1+y^2}+g(y)$

$\dfrac{\partial f}{\partial y}=\dfrac{x^2(1+y^2)-2y(x^2y)}{(1+y^2)^2}+\dfrac{\partial g}{\partial y}=\dfrac{x^2-x^2y^2}{(1+y^2)^2}+\dfrac{\partial g}{\partial y}=\dfrac{x^2-2y-x^2y^2}{(1+y^2)^2}$

therefore $\dfrac{\partial g}{\partial y}=-\dfrac{2y}{(1+y^2)^2}$, and $g(y)=\dfrac{1}{1+y^2}+C$

then $f(x,y)=\dfrac{1+x^2y}{1+y^2}+C$

1 mark

$\displaystyle\int_{(-1,2)}^{(3,0)} F\,dr=f(3,0)-f(-1,2)=1-\frac{3}{5}=\boxed{\frac{2}{5}}$