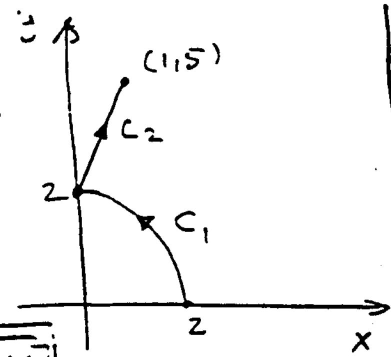
5. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $F=(1,x^2+y)$, and the curve C consists of the part of the circle $x^2+y^2=4$ from (2,0) to (0,2), followed by the line segment from (0,2) to (1,5). (15 marks)

$$\int_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr$$



$$C_1: \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases} \quad 0 \le t \le \frac{\pi}{2}$$

$$\int_{C_1} F\cdot dr = \int_0^{\pi/2} (1, 4\cos^2 t + 2\sin t)\cdot(-2\sin t, 2\cos t)\,dt$$

$$= \int_0^{\pi/2} (-2\sin t + 8\cos^3 t + 4\sin t\cos t)\,dt$$

4 marks

$$= \int_0^{\pi/2} (-2\sin t + 8\cos t - 8\sin^2 t\cos t + 4\sin t\cos t)\,dt$$

$$= -2\cos t + 8\sin t - \frac{8}{3}\sin^3 t + 2\sin^2 t\Big]_0^{\pi/2}$$

$$= 8 - \frac{8}{3} + 2 + 2 = \frac{28}{3}$$

6 marks

$$C_2: \begin{cases} x = t \\ y = 2+3t \end{cases} \quad 0 \le t \le 1$$

$$\int_{C_2} F\cdot dr = \int_0^1 (1, t^2+2+3t)\cdot(1,3)\,dt$$

$$= \int_0^1 (7+9t+3t^2)\,dt = 7t + \frac{9t^2}{2} + t^3\Big]_0^1$$

$$= 8 + \frac{9}{2} = \frac{25}{2}$$

5 marks

$$\int_C F\cdot dr = \frac{28}{3} + \frac{25}{2} = \frac{56+75}{6} = \boxed{\frac{131}{6}}$$

6. a) Consider the vector field $F(x,y)=(y^2\cos x-y^2e^{xy}, 2y+2y\sin x-(1+xy)e^{xy})$. Decide whether or not $F(x,y)$ has a potential function, and find a potential function if there is one. (7 marks)

b) Evaluate $\int_C 3dx+2yzdy+(1+y^2)dz$, where C is the curve parametrized by $r(t)=(\frac{2}{1+\sin t}, \frac{2}{1+\cos t}, -\sin t)$, $0\le t\le\pi/2$. (8 marks)

3 marks

a) $\frac{\partial u}{\partial y} = 2y\cos x - 2ye^{xy} - xy^2e^{xy}$

$\frac{\partial v}{\partial x} = 2y\cos x - ye^{xy} - y(1+xy)e^{xy} = 2y\cos x - 2ye^{xy} - xy^2e^{xy}$

$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ Therefore, $F(x,y)$ has a potential function

4 marks

$\int (y^2\cos x - y^2e^{xy})dx = y^2\sin x - ye^{xy} + h(y)$

Then $f(x,y) = y^2\sin x - ye^{xy} + h(y)$, and

$\frac{\partial f}{\partial y} = 2y\sin x - e^{xy} - xye^{xy} + h'(y)$

Therefore $h'(y) = 2y$, and $\boxed{f(x,y) = y^2\sin x - ye^{xy} + y^2 + C}$

b) Note that $f(x,y,z) = 3x + y^2z + z$ implies $\nabla f(x,y,z) = (3, 2yz, y^2+1)$, therefore

3 marks

$\int_C 3dx + 2yzdy + (1+y^2)dz = f(B) - f(A)$ where A and B are the endpoints of the curve C.

3 marks

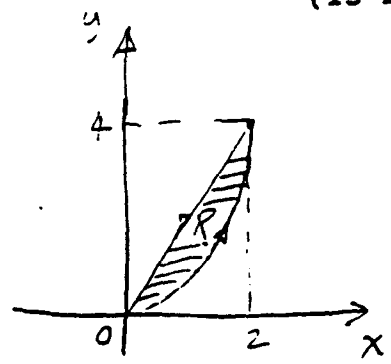
$A = (2,1,0)$ and $B = (1,2,-1)$

2 marks

$\int 3dx + 2yzdy + (1+y^2)dz = 3-4-1-6 = \boxed{-8}$

7. Evaluate $\int_C (y+\ln(1+x^3))\,dx+x^2dy$, where the curve C is the boundary of the region enclosed between the parabola $y=x^2$ and the line $y=2x$, traversed once, in the counterclockwise direction. (15 marks)

Using Green's Theorem:



$0 \leq x \leq 2$

$x^2 \leq y \leq 2x$

$$\int_C (y+\ln(1+x^3))\,dx + x^2\,dy$$

$$= \iint_R (2x-1)\,dA$$

$$= \int_0^2 \int_{x^2}^{2x} (2x-1)\,dy\,dx$$

$$= \int_0^2 (2xy-y)\Big]_{x^2}^{2x} dx = \int_0^2 (4x^2-2x-2x^3+x^2)\,dx$$

$$= \frac{5x^3}{3} - x^2 - \frac{x^4}{2}\Big]_0^2 = \frac{40}{3} - 4 - 8 = \boxed{\frac{4}{3}}$$