UNIVERSITY OF TORONTO

DEPARTMENT OF MATHEMATICS

MAT 235 Y - TEST #3 MARCH 21, 1995

NAME: MODEL SOLUTIONS STUDENT No.: _____

(Family name. Please PRINT.) (Given name.)

+ MARKING SCHEME

INSTRUCTIONS: Show and explain all your work in all questions. Give your answers in the space provided. Use both sides of paper, if necessary. Do not tear out any pages. No calculators or other aids are permitted. Time allowed: 100 minutes

1. Evaluate $\iint_R xy\,dA$, where $R=\{(x,y): 1\le x^2+y^2\le 4,\ 0\le x\le y\}$. (10 marks)

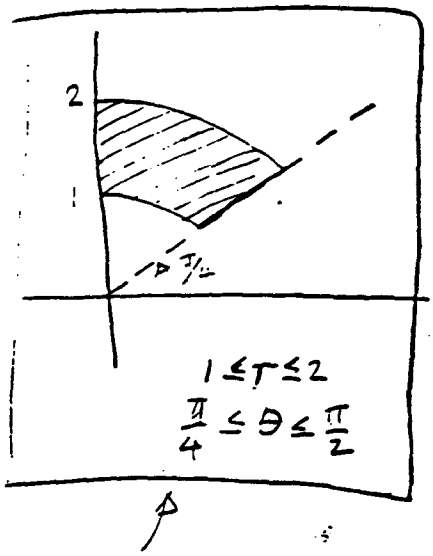
$$\iint_R xy\,dA = \int_1^2 \int_{\pi/4}^{\pi/2} r^3 \sin\theta\cos\theta\, d\theta\, dr$$

3 marks

$$= \left[\frac{r^4}{4}\right]_1^2 \left[\frac{1}{2}\sin^2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(\frac{15}{4}\right)\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = \boxed{\frac{15}{16}}$$

4 marks



$1 \le r \le 2$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

3 marks

2. Find the surface area of the part of the sphere $x^2+y^2+z^2=4$ lying above the plane $z=1$. (15 marks)

$$A = 4\iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

Where $\frac{\partial z}{\partial x} = -\frac{x}{z}$ $\qquad$ $\frac{\partial z}{\partial y} = -\frac{y}{z}$

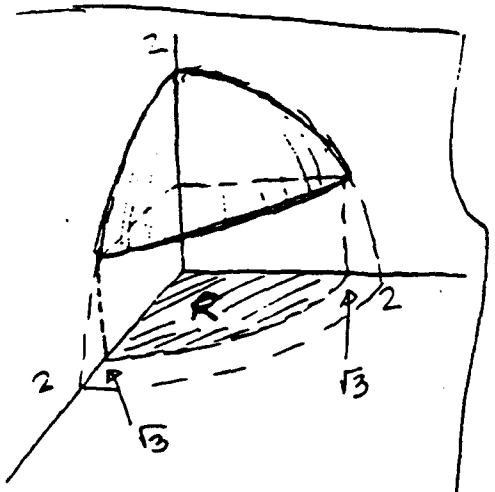
3 marks

$$A = 4\iint_R \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\,dA$$

$$= 4\iint_R \sqrt{\frac{4}{z^2}}\,dA$$

$$= 4\iint_R \frac{2}{z}\,dA = 4\iint_R \frac{2}{\sqrt{4-x^2-y^2}}\,dA$$

marks



$0 \le r \le \sqrt{3}$

$0 \le \theta \le \frac{\pi}{2}$

5 marks

$$= 4\int_0^{\sqrt{3}}\int_0^{\pi/2} \frac{2r}{\sqrt{4-r^2}}\,d\theta\,dr = 4\left[\theta\right]_0^{\pi/2}\left[-2\sqrt{4-r^2}\right]_0^{\sqrt{3}}$$

$$= 4\left[\frac{\pi}{2}\right]\left[2\right] = \boxed{4\pi}$$

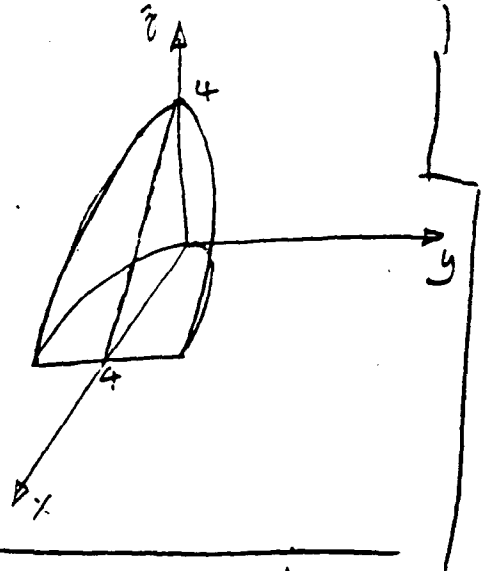
4 marks

3. Find the volume of the region bounded by the surface $x=y^2$ and the planes $z=0$ and $x+z=4$. (15 marks)

$R: \quad -2 \le y \le 2$

$y^2 \le x \le 4$

$0 \le z \le 4-x$



7 marks

$$V = 2\int_0^2 \int_{y^2}^{4} \int_0^{4-x} dz\,dx\,dy$$

$$= 2\int_0^2 \int_{y^2}^{4} (4-x)\,dx\,dy$$

$$= 2\int_0^2 \left[4x - \frac{x^2}{2}\right]_{y^2}^{4} dy = 2\int_0^2 \left[16 - 2 - 4y^2 + \frac{y^4}{2}\right] dy$$

$$= 2\int_0^2 \left[8 - 4y^2 + \frac{y^4}{2}\right] dy$$

$$= 2\left[8y - \frac{4y^3}{3} + \frac{y^5}{10}\right]_0^2$$

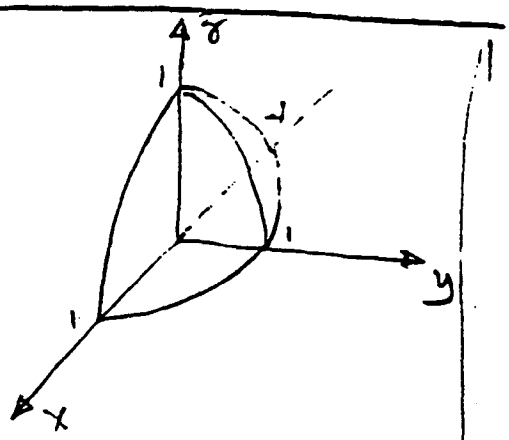
$$= 2\left[16 - \frac{32}{3} + \frac{32}{10}\right] = 32\left[1 - \frac{2}{3} + \frac{2}{10}\right]$$

$$= 32\left[\frac{30-20+6}{30}\right] = \boxed{\frac{256}{15}}$$

8 marks

4. Evaluate the integral $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} ye^{(x^2+y^2+z^2)^2}\,dz\,dy\,dx$, by changing to spherical coordinates. (15 marks)

The region $\begin{cases} -1 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \\ 0 \le z \le \sqrt{1-x^2-y^2} \end{cases}$



In spherical coordinates:

$\begin{cases} 0 \le \rho \le 1 \\ 0 \le \theta \le \pi \\ 0 \le \phi \le \frac{\pi}{2} \end{cases}$

$$\int_0^1\int_0^{\pi}\int_0^{\pi/2} (\rho \sin\phi \sin\theta)\, e^{\rho^4}\, (\rho^2 \sin\phi)\, d\phi\, d\theta\, d\rho$$

7 marks

3 marks

$$= \int_0^1\int_0^{\pi}\int_0^{\pi/2} \rho^3 e^{\rho^4} \sin\theta \sin^2\phi\, d\phi\, d\theta\, d\rho$$

$$= \left[\frac{1}{4}e^{\rho^4}\right]_0^1 \left[-\cos\theta\right]_0^{\pi} \left[\frac{1}{2}\phi - \frac{1}{2}\sin(2\phi)\right]_0^{\pi/2}$$

$$= \frac{1}{4}(e-1)(2)\left(\frac{\pi}{4}\right) = \boxed{\frac{\pi(e-1)}{8}}$$

3 marks