TEST #3
Feb 28, 1994

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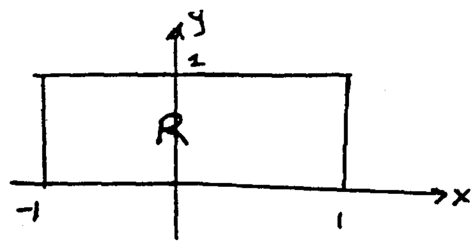
1. Evaluate

$$\iint_R \frac{x+1}{y+1}\,dx\,dy$$

where R is the region $|x| \le 1$, $0 \le y \le 1$.



$$\iint_R \frac{x+1}{y+1}\,dx\,dy$$

$$= \int_0^1 \int_{-1}^1 \frac{x+1}{y+1}\,dx\,dy$$

$$= \int_0^1 \frac{dy}{y+1} \int_{-1}^1 x+1\,dx$$

$$= \left(\ln|y+1|\Big|_0^1\right)\left(\frac{x^2}{2}+x\Big|_{-1}^1\right)$$

$$= (\ln 2)\cdot\left(\tfrac{1}{2}+1-\tfrac{1}{2}+1\right)$$

$$= 2\ln 2 \;//$$

2. Evaluate $\iint_R (x^3 - 5y + 4)\,dx\,dy$ where R is the unit disk centered at 0.

$$\iint_R (x^3 - 5y+4)\,dx\,dy$$

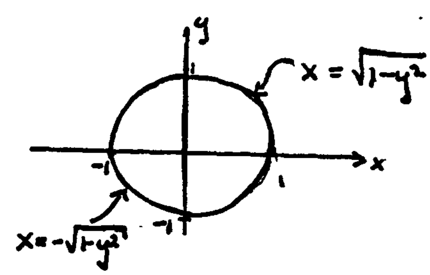
$$= \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^3-5y+4)\,dx\,dy$$

$$= \int_{-1}^1 \frac{x^4}{4} - (5y+4)x \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\,dy$$

$$= \int_{-1}^1 \frac{(1-y^2)^2}{4} - \frac{(1-y^2)^2}{4} - 10y\sqrt{1-y^2} + 8\sqrt{1-y^2}\,dy$$

$$= 5\int_{-1}^1 -2y\sqrt{1-y^2}\,dy + 8\int_{-1}^1 \sqrt{1-y^2}\,dy$$

$$= \underbrace{5\cdot\frac{2}{3}\,(1-y^2)^{3/2}\Big|_{-1}^1}_{=0} + 8\int_{-1}^1 \sqrt{1-y^2}\,dy$$



Let $y=\sin\theta$
$dy=\cos\theta\,d\theta$

$$\stackrel{\downarrow}{=} 8\int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$= 8\int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta$$

$2\cos^2\theta - 1 = \cos 2\theta$

$$\stackrel{\downarrow}{=} 8\int_{-\pi/2}^{\pi/2} \left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)d\theta$$

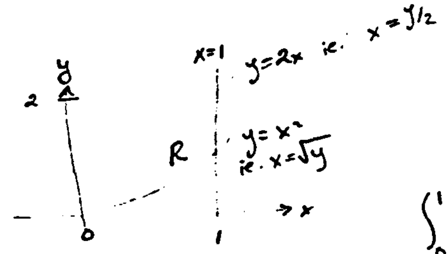
$$= 8\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Big|_{-\pi/2}^{\pi/2}$$

3. Express

$$\int_0^1 \int_{x^2}^{2x} f(x,y)\,dy\,dx$$

as an iterated integral in the opposite order.

Let R be the region given by $x^2 \le y \le 2x$, $0 \le x \le 1$

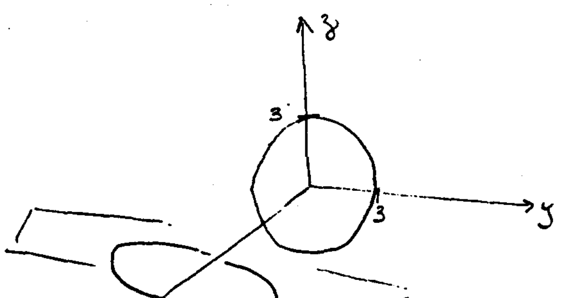


Then

$$\int_0^1 \int_{x^2}^{2x} f(x,y)\,dy\,dx$$

$$= \int_0^2 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy \quad //$$

4. Find the volume of the region inside the cylinder $y^2 + z^2 = 9$ and between the yz-plane and the plane $x + y - 2z = 32$.



$x+y-2z=32$

ie. $x = 32 - y + 2z$

Let C be the circle $y^2+z^2=9$ in the yz-plane.

$$V = \iint_C \int_0^{32-y+2z} dx\,dC$$

Let $y = r\cos\theta$, $z = r\sin\theta$ $\begin{pmatrix} 0 \le r \le 3 \\ 0 \le \theta < 2\pi \end{pmatrix}$

jacobian

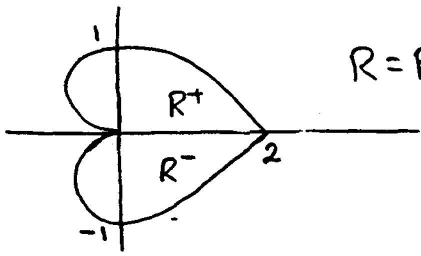
$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$V = \int_0^{2\pi} \int_0^3 (32 - r\cos\theta + 2r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} 16r^2 - \frac{r^3}{3}\cos\theta + r^2\sin\theta \Big|_0^3 \,d\theta$$

$$= \int_0^{2\pi} 144 - 3\cos\theta + 9\sin\theta \,d\theta$$

5. Find the volume of the region lying over the cardioid $r = 1 + \cos\theta$ and under the graph of $z = r$.



$R = R^+ \cup R^-$

$$V = \int_R \int_0^r dz \, dR$$

$$= \int_R r \cdot r \, dr \, d\theta$$

$$= 2 \int_{R^+} r^2 \, dr \, d\theta$$

$$= 2 \int_0^{\pi} \int_0^{1+\cos\theta} r^2 \, dr \, d\theta$$

$$= \frac{2}{3} \int_0^{\pi} (1+\cos\theta)^3 \, d\theta$$

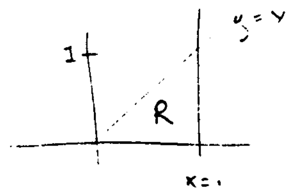
$$= \frac{2}{3} \int_0^{\pi} (1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta) \, d\theta$$

$$= \frac{2}{3} \left[\pi + \underbrace{\left(3\sin\theta \Big|_0^{\pi}\right)}_{0} + 3\underbrace{\left(\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]\Big|_0^{\pi}\right)}_{\pi/2} + \int_0^{\pi} (1 - \sin^2\theta)(\cos\theta \, d\theta) \right.$$

$$= \frac{2}{3} \left[\frac{5\pi}{2} + \underbrace{\int_0^{\pi} \cos\theta \, d\theta}_{=0} - \underbrace{\int_0^{\pi} \sin^2\theta \cos\theta \, d\theta}_{= \frac{1}{3}\sin^3\theta \Big|_0^{\pi} = 0} \right]$$

$$= \frac{5\pi}{3}$$

6. Find the area of the surface $z = 2 + x^2$ lying over the triangle $0 \le y \le x$, $0 \le x \le 1$.



Let $f(x,y) = 2 + x^2$

then $f_x = 2x$

$f_y = 0$.

so $\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1+4x}$

$$S.A. = \int_0^1 \int_0^x \sqrt{1 + f_x^2 + f_y^2}\, dy\, dx$$

$$= \int_0^1 \int_0^x \sqrt{1 + 4x^2}\, dy\, dx$$

$$= \int_0^1 x\sqrt{1+4x^2}\, dx$$

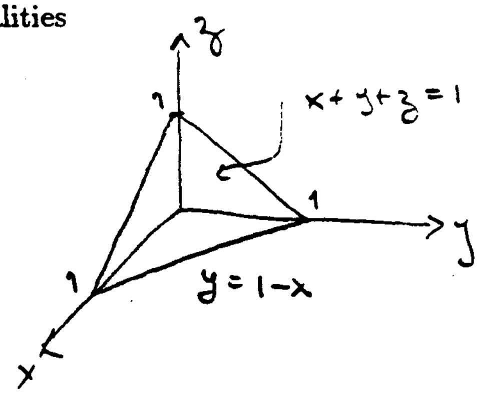
$$= \frac{1}{8} \cdot \frac{2}{3} \left(1 + 4x^2\right)^{3/2} \Big|_0^1$$

$$= \frac{1}{12}\left(5^{3/2} - 1\right) \quad //$$

7. Find the centroid of the tetrahedron given by the inequalities

Let R be this tetrahedron.

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$
$$x+y+z \leq 1.$$



$$M_{xy} = \iiint_R z\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2}\,dy\,dx$$

$$= \frac{-1}{6} \int_0^1 (1-x-y)^3 \Big|_{y=0}^{y=1-x} dx$$

$$= 1/6 \int_0^1 (1-x)^3\,dx$$

$$= 1/6 \cdot (-1/4) \quad (1-x^4)\Big|_0^1 = 1/24$$

$$M = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} 1-x-y\,dy\,dx$$

$$= \int_0^1 \left(-\tfrac{1}{2}\right)(1-x-y)^2 \Big|_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 (1-x)^2\,dx$$

$$= \frac{-1}{2}\,\frac{(1-x)^3}{3}\Big|_0^1 = \frac{1}{6}.$$

So $\bar{z} = \dfrac{M_{xy}}{M} = \dfrac{1/24}{1/6} = \dfrac{1}{4}.$

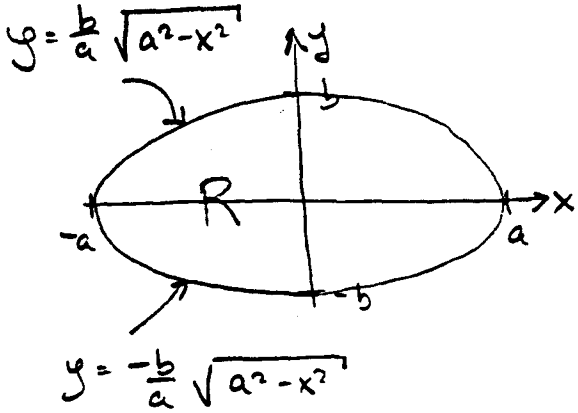
By symmetry, $\bar{x} = \bar{y} = \bar{z} = 1/4$ and the centroid is

$(1/4, 1/4, 1/4)$. //

8. Let a and b be positive numbers and consider the region R given by

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &\le 1 \\ x &\le 0. \end{aligned}$$

(So R is the left half of an ellipse.) Find the centroid of R in terms of a and b.



Let centroid be $(\bar{x}, \bar{y})$. By symmetry about the x-axis, $\bar{y} = 0$.

To calculate $\bar{x}$:

$$M_y = \int_{-a}^{0} \int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} x \, dy \, dx$$

$$= \frac{2b}{a} \int_{-a}^{0} x\sqrt{a^2-x^2}\, dx$$

$$= \frac{2b}{a} \cdot \frac{2}{3} \cdot -\frac{1}{2} \cdot (a^2-x^2)^{3/2} \Big|_{-a}^{0}$$

$$= \frac{-2b}{3a} \cdot a^3$$

$$= -\frac{2ba^2}{3}$$

Also $M = \iint_R dy\,dx = \iint ab\, dy'\, dx' = \frac{ab\pi}{2}$

set $x' = \frac{x}{a}$, $y' = \frac{y}{b}$

left ½ of unit circle

So $\bar{x} = M_y / M = -\frac{2ba^2}{3} \cdot \frac{2}{ab\pi} = \frac{-4a}{3\pi}$

So centroid is $\left(-\frac{4a}{3\pi}, 0\right)$. //