MODEL SOLUTIONS

① a)

$$\boxed{\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{y\sin(y+z)+ze^{xz}}{xy\cos(y+z)+xe^{xz}}\\ \frac{\partial z}{\partial y} &= -\frac{x\sin(y+z)+xy\cos(y+z)}{xy\cos(y+z)+xe^{xz}}\end{aligned}}$$

b) $$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$= 4u^2(x+3y)^3 - 12(x+3y)^3$$

At $(u,v)=(-1,2)$, $(x,y)=(2,-4)$, and

$$\frac{\partial f}{\partial v} = 16(-10)^3 - 12(-10)^3 = \boxed{-4000}$$

② a) $f_x = e^{x-2y}$ $\qquad f_y = -2e^{x-2y}$

Take $(x_0,y_0)=(2,1)$; $(h,k)=(0.01, 0.02)$, then

$$f(2.01, 1.02) \approx e^0 + (0.01)e^0 + (0.02)(-2e^0)$$

$$= 1 + 0.01 - 0.04 = \boxed{0.97}$$

b) $\nabla f = (2xy,\ x^2-3y^2z,\ -y^3+2z)$

$\nabla f(1,1,1) = (2,-2,1)$ $\qquad \|\nabla f\| = 3$

Max. possible value: $\boxed{3}$

(3) $u_x = 3x(x^2+y^2)^{1/2}$ $\quad u_{xx} = 3\left[(x^2+y^2)^{1/2} + x^2(x^2+y^2)^{-1/2}\right]$

Similarly: $\quad u_{yy} = 3\left[(x^2+y^2)^{1/2} + y^2(x^2+y^2)^{-1/2}\right]$

$$u_{xx} + u_{yy} = 3\left[2(x^2+y^2)^{1/2} + (x^2+y^2)(x^2+y^2)^{-1/2}\right]$$

$$= 9(x^2+y^2)^{1/2} = 9u^{1/3} \qquad \boxed{K=9}$$

(4) a) $f_x = 3x^2 - 15 + 3y^2 \qquad f_y = 6xy + 3y^2$

From $f_y = 0$: $\quad 3y(2x+y) = 0 \quad y=0$ or $y=-2x$

If $y=0$, then $x = \pm\sqrt{5}$

If $y=-2x$, then $x = \pm 1$

$\boxed{\text{Critical points: } (\pm\sqrt{5}, 0) \text{ and } (\pm 1, \mp 2)}$

$f_{xx} = 6x \qquad f_{xy} = 6y \qquad f_{yy} = 6x + 6y$

At $(\sqrt{5}, 0)$: $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$

At $(-\sqrt{5}, 0)$: $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$

At $(1,-2)$: $f_{xx}f_{yy} - f_{xy}^2 < 0$

At $(-1,2)$: $f_{xx}f_{yy} - f_{xy}^2 < 0$

$$\boxed{\begin{array}{l}\text{Local min at } (\sqrt{5}, 0) \\ \text{Local max at } (-\sqrt{5}, 0) \\ \text{Saddle points at } (\pm 1, \mp 2)\end{array}}$$

b) For $\nabla f = \lambda \nabla g$:

$-\frac{1}{x^2} = 8\lambda x$, $-\frac{2}{y^2} = 2\lambda y$, $-\frac{2}{z^2} = 16\lambda z$

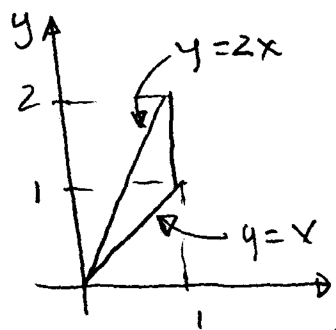
Then $y = 2x$, $z = x$, and $4x^2 + 4x^2 + 8x^2 = 1$

$x = \frac{1}{4}$, $y = \frac{1}{2}$, $z = \frac{1}{4}$

$f(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}) = 4 + 4 + 8$

Min value of f: $\boxed{16}$

5 a)



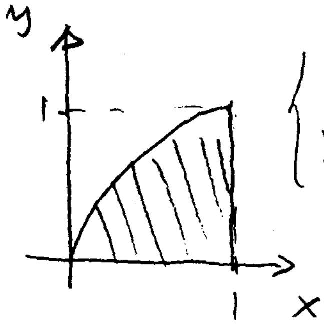
$$V = \int_0^1 \int_x^{2x} (1 + x + xy)\, dy\, dx$$

$$= \int_0^1 \Big[(1+x)y + \frac{1}{2} x y^2\Big]_x^{2x} dx$$

$$= \int_0^1 \Big[x + x^2 + \frac{3}{2}x^3\Big] dx = 1 + \frac{1}{3} + \frac{3}{8}$$

$$\boxed{V = \frac{41}{24}}$$

b)



$\begin{cases} 0 \le y \le 1 \\ y^2 \le x \le 1 \end{cases}$ or $\begin{cases} 0 \le x \le 1 \\ 0 \le y \le \sqrt{x} \end{cases}$

$$\int_0^1 \int_{y^2}^1 \frac{8y^5}{1+x^4}\, dx\, dy = \int_0^1 \int_0^{\sqrt{x}} \frac{8y^5}{1+x^4}\, dy\, dx = \int_0^1 \frac{4}{3} \frac{x^3}{1+x^4}\, dx$$

$$= \frac{1}{3} \ln(1+x^4)\Big]_0^1 = \boxed{\frac{1}{3} \ln 2}$$