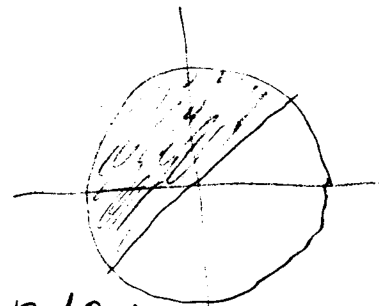
4. (7 marks) Find the volume of the region lying over the semidisk $x^2 + y^2 \le 4$, $y \ge x$, and under the paraboloid $z = x^2 + y^2$.

We convert to polar coordinates.

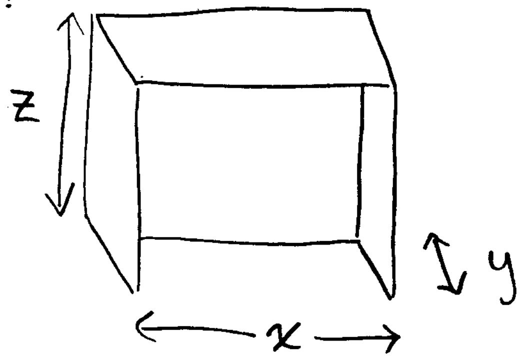


$$\iint_R (x^2+y^2)\,dx\,dy = \int_0^2\int_{\pi/4}^{5\pi/4} (x^2+y^2)\,r\,d\theta\,dr$$

$$= \int_0^2\int_{\pi/4}^{5\pi/4} r^3\,d\theta\,dr = \int_0^2 r^3\int_{\pi/4}^{5\pi/4} d\theta\,dr$$

$$= \int_0^2 r^3\cdot\pi\,dr = \pi\int_0^2 r^3\,dr = \pi\left[\frac{r^4}{4}\right]_0^2 = 4\pi.$$

5. (8 marks) A bus shelter is made in the shape of a rectangular box with no front and no bottom. The material for the roof costs \$5 per m^2 and the material for the other three sides costs \$4 per m^2. What are the dimensions of the least expensive shelter with a volume of 20 m^3?



$xyz = 20$

Area of roof $= xy$

Area of other three sides $= xz + 2yz$

so cost of shelter is $5xy + 4(xz + 2yz)$

$= 5xy + 4xz + 8yz$

Since $z = \frac{20}{xy}$, this can also be written $5xy + 4x\left(\frac{20}{xy}\right) + 8y\left(\frac{20}{xy}\right)$

$= 5xy + \frac{80}{y} + \frac{160}{x}$. Call this $f(x,y)$. We wish to minimize $f(x,y)$ so we find the critical points.

$\frac{\partial f}{\partial x} = 5y - \frac{160}{x^2}$, $\frac{\partial f}{\partial y} = 5x - \frac{80}{y^2}$.

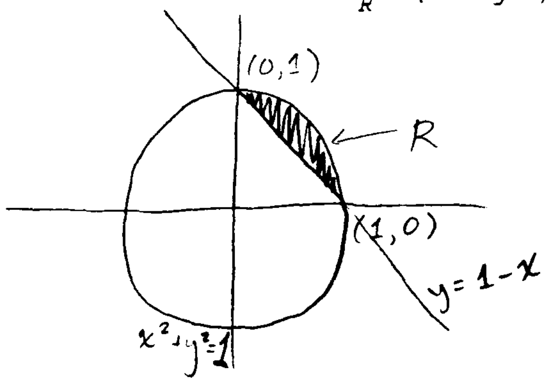
Solving $5y - \frac{160}{x^2} = 0 = 5x - \frac{80}{y^2}$ gives us

$5y = \frac{160}{x^2}$, $5x = \frac{80}{y^2} \Rightarrow 5xy = \frac{160}{x}$, $5xy = \frac{80}{y}$

$\Rightarrow \frac{160}{x} = \frac{80}{y} \Rightarrow x = 2y \Rightarrow 5(2y) = \frac{80}{y^2} \Rightarrow 10y^3 = 80 \Rightarrow y^3 = 8$

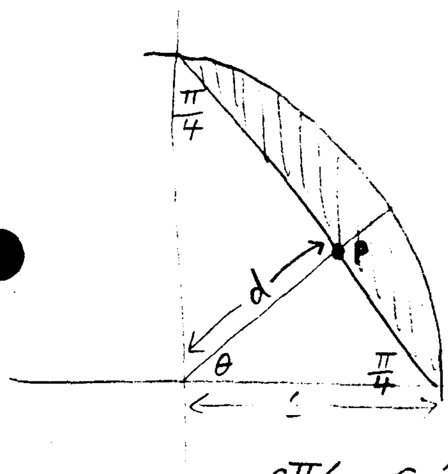
$\Rightarrow y = 2$, $x = 4$, $z = \frac{5}{2}$.

6. (6 marks) Evaluate $\iint_R \frac{1}{(x^2+y^2)^2}\,dx\,dy$ where $R=\{(x,y): x^2+y^2 \le 1,\ x+y \ge 1\}$.



We convert to polar coordinate: The region R can be expressed using polar coordinates as

$\{ (r,\theta): 0 \le \theta \le \frac{\pi}{2},\ d \le r \le 1 \}$ where d is as illustrated



But the polar coordinates of the point P are $(d\cos\theta, d\sin\theta)$, and P is on the line $x+y=$ So we have $d\cos\theta + d\sin\theta = 1$, so $d = \frac{1}{\cos\theta+\sin\theta}$. Thus the integral we wish to evaluate is equal to

$$\int_0^{\pi/2}\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} \frac{1}{(x^2+y^2)^2}\, r\,dr\,d\theta = \int_0^{\pi/2}\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r^{-3}\,dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[\frac{1}{r^2}\right]_{\frac{1}{\cos\theta+\sin\theta}}^{1} d\theta = -\frac{1}{2}\int_0^{\pi/2}\left(1-(\cos\theta+\sin\theta)^2\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 2\sin\theta\cos\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2}\sin 2\theta\,d\theta$$ Subst. $u=2\theta$

$$= \frac{1}{4}\int^{\pi} \sin u\,du = \frac{1}{4}(2) = \frac{1}{2}.$$

7. (-- marks) Find the surface area of the part of the plane $x + 3y - z = 0$ lying inside the cylinder $x^2 + y^2 = 9$.

Let $f(x,y) = x + 3y$ and let R be the region $\{(x,y) : x^2 + y^2 \le 9\}$. Then we wish to find the surface area of that portion of the surface $z = f(x,y)$ which lies over the region R.

So the surface area is

$$\iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$$

$$= \iint_R \sqrt{1 + 1^2 + 3^2}\, dx\, dy = \sqrt{11} \iint_R dx\, dy$$

$$= \sqrt{11} \cdot (\text{area of } R) = \sqrt{11} \cdot 9\pi = 9\sqrt{11}\,\pi.$$