UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS

MAT 235 Y - CALCULUS FOR PHYSICAL AND LIFE SCIENCES
FALL-WINTER 1995-96

TEST #2. JANUARY 23, 1996

NAME: ________________, __________ STUDENT No: __________
(Family name. Please PRINT.) (Given name.)

INSTRUCTIONS: This test consists of six questions. The value of each question is indicated (in brackets) by the question number. Total marks: 45.
Show all your work in all questions. Give your answers in the space provided. Use both sides of the paper, if necessary. Do not tear out any pages.
No calculators or any other aids are permitted. This test is worth 20% of your course grade. Keep your student card visible on your table. Time allowed: 2 hours.

1. (7 marks) Find the critical points of the function $f(x,y) = x^3 + y^2 - 6xy + 6x + 3y$, and use the Second Derivative Test to determine whether they are local maxima, local minima or saddle points.

$$\frac{\partial f}{\partial x} = 3x^2 - 6y + 6 \qquad \frac{\partial f}{\partial y} = 2y - 6x + 3$$

If both partial derivatives are 0, we have $2y - 6x + 3 = 0$
$\Rightarrow 2y = 6x - 3 \Rightarrow y = 3x - \frac{3}{2} \Rightarrow 3x^2 - 6\left(3x - \frac{3}{2}\right) + 6 = 0$
$\Rightarrow 3x^2 - 15x + 15 = 0 \Rightarrow x^2 - 6x + 5 = 0$
$\Rightarrow x = 5$ or $x = 1$. If $x = 5$ then $y = 3(5) - \frac{3}{2} = \frac{27}{2}$
and if $x = 1$ then $y = 3(1) - \frac{3}{2} = \frac{3}{2}$.
Furthermore, the partial derivatives both exist everywhere, so the only critical points are those where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, i.e. the two points $\left(5, \frac{27}{2}\right)$ and $\left(1, \frac{3}{2}\right)$.

Now, $\frac{\partial^2 f}{\partial x^2} = 6x$, $\frac{\partial^2 f}{\partial x \partial y} = -6$, $\frac{\partial^2 f}{\partial y^2} = 2$.

At the critical point $\left(5, \frac{27}{2}\right)$, we have $f_{xy}^2 - f_{xx} f_{yy} = (-6)^2 - (30)(2)$ which is negative. Since f_{xx} is positive, f has a local min. at $\left(5, \frac{27}{2}\right)$.
At the critical point $\left(1, \frac{3}{2}\right)$, we have $f_{xy}^2 - f_{xx} f_{yy} = (-6)^2 - (6)(2)$ which is positive, so f has a saddle point at $\left(1, \frac{3}{2}\right)$.

2. (7 marks) Find the maximum and minimum values of the function $f(x,y) = 5 + x - 2y$, on the ellipse $x^2 + 4y^2 = 2$.

We want to find the extreme values of the function $f(x,y) = 5 + x - 2y$ subject to the constraint $g(x,y) = 0$ where $g(x,y) = x^2 + 4y^2 - 2$.

Thus we can use the method of Lagrange multipliers.

$f_x = 1 \quad f_y = -2 \quad g_x = 2x \quad g_y = 8y$

so we have the three equations

$$x^2 + 4y^2 - 2 = 0$$
$$1 = \lambda \cdot 2x$$
$$-2 = \lambda \cdot 8y$$

This gives $x = \frac{1}{2\lambda}$, $y = \frac{-1}{4\lambda} \Rightarrow \left(\frac{1}{2\lambda}\right)^2 + 4\left(\frac{-1}{4\lambda}\right)^2 - 2 = 0$

$\Rightarrow \lambda^2 = \frac{1}{4}$, so $\lambda = \frac{1}{2}$ or $-\frac{1}{2}$. If $\lambda = \frac{1}{2}$ then $x = 1$ and $y = -\frac{1}{2}$. If $\lambda = \frac{-1}{2}$ then $x = -1$ and $y = \frac{1}{2}$

Now $f(1, -\frac{1}{2}) = 5 + 1 + 1 = 7$

and $f(-1, \frac{1}{2}) = 5 - 1 - 1 = 3$

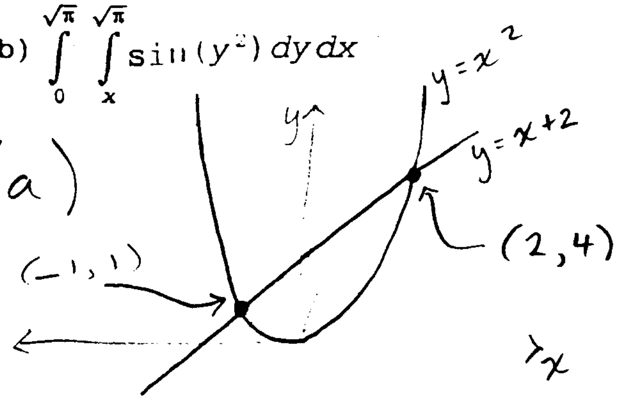
Thus the max. and min. values of f subject to the given constraint are 7 and 3 respectively, which are attained at $(1, -\frac{1}{2})$ and $(-1, \frac{1}{2})$ respectively.

3. (10 marks) Evaluate each of the following integrals:

a) $\iint_R x\,dA$ where R is the region enclosed between the parabola $y=x^2$ and the line $y=x+2$.

b) $\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \sin(y^2)\,dy\,dx$

(a)



$$\iint_R x\,dA = \int_{-1}^{2}\int_{x^2}^{x+2} x\,dy\,dx$$

$$= \int_{-1}^{2} x \int_{x^2}^{x+2} dy\,dx = \int_{-1}^{2} x\left(x+2-x^2\right)dx$$

$$= \int_{-1}^{2}(x^2+2x-x^3)\,dx = \left[\frac{x^3}{3}+x^2-\frac{x^4}{4}\right]_{-1}^{2} = \frac{9}{4}$$

(b) $$\int_0^{\sqrt{\pi}}\int_x^{\sqrt{\pi}} \sin(y^2)\,dy\,dx = \int_0^{\sqrt{\pi}}\int_0^{y} \sin(y^2)\,dx\,dy$$

$$= \int_0^{\sqrt{\pi}} \sin(y^2)\int_0^{y} dx\,dy = \int_0^{\sqrt{\pi}} y\sin(y^2)\,dy$$

$$= \left[-\frac{1}{2}\cos(y^2)\right]_0^{\sqrt{\pi}} = \frac{1}{2}-\left(-\frac{1}{2}\right) = 1.$$