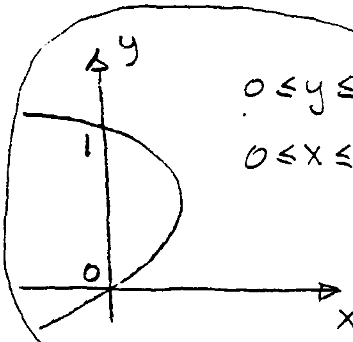
6\. a) Integrate the function $f(x,y)=y$ over the region enclosed between the parabola $x=y-y^2$ and the y-axis. (4 marks)

b) Integrate the function $f(x,y)=y|x-3|$ over the rectangle $0 \le x \le 4$, $1 \le y \le 2$. (4 marks)

a)



$0 \le y \le 1$

$0 \le x \le y-y^2$

2 marks

$$\int_0^1\int_0^{y-y^2} y\,dx\,dy = \int_0^1 xy\Big]_0^{y-y^2} dy$$

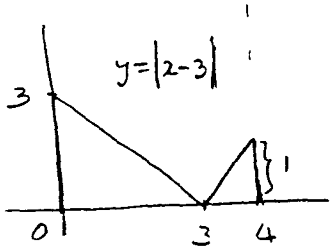
$$= \int_0^1 (y^2-y^3)\,dy = \frac{y^3}{3} - \frac{y^4}{4}\Big]_0^1 = \frac{1}{3} - \frac{1}{4}$$

$$= \boxed{\frac{1}{12}}$$

2 marks

b) $$\int_0^4\int_1^2 y|x-3|\,dy\,dx = \int_0^4 |x-3|\,dx \int_1^2 y\,dy$$

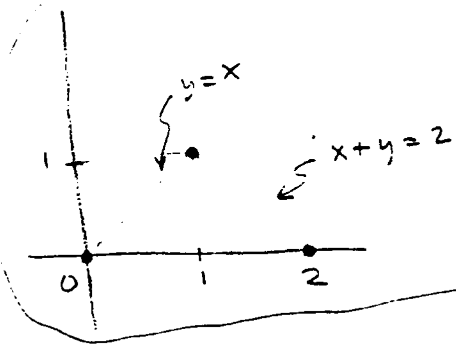
4 marks



$$= \left(\frac{9}{2}+\frac{1}{2}\right)\left(\frac{y^2}{2}\Big]_1^2\right) = 5\left[2-\frac{1}{2}\right] = \boxed{\frac{15}{2}}$$

7. Find the volume of the region above the triangle with vertices (0,0), (1,1), and (2,0), and under the paraboloid $z = 9 - x^2 - y^2$. (6 marks)

2 marks



$0 \le y \le 1$

$y \le x \le 2-y$

1 mark

$$V = \int_0^1 \int_y^{2-y} (9-x^2-y^2)\,dx\,dy = \int_0^1 (9-y^2)x - \frac{x^3}{3}\Big]_y^{y+2}\,dy$$

2 marks

$$= \int_0^1 \Big[(9-y^2)(y+2) - \frac{(y+2)^3}{3} - (9-y^2)y + \frac{y^3}{3}\Big]_0^1\,dy$$

$$= \int_0^1 \Big[\cancel{y^3} - 2y^2 + \cancel{9y} + 18 - \frac{y^3+6y^2+12y+8}{3} - \cancel{9y} + \cancel{y^3} + \cancel{\frac{y^3}{3}}\Big]\,dy$$

$$= \int_0^1 \Big(-4y^2 - 4y + \frac{46}{3}\Big)\,dy = -\frac{4y^3}{3} - 2y^2 + \frac{46}{3}y\Big]_0^1$$

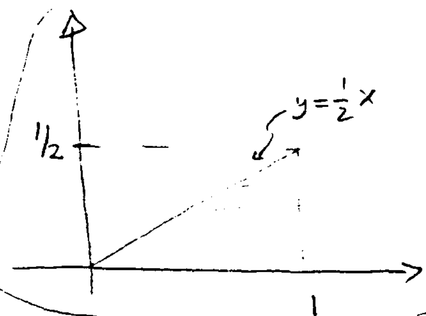
$$= -\frac{4}{3} - 2 + \frac{46}{3} = \boxed{12}$$

8. a) Evaluate $\int_0^{1/2}\int_{2y}^{1} \frac{2e^{2y}}{(e^x-x)^2}\,dx\,dy$. (3 marks)

b) Let f(x) be a continuous function over the interval [a,b], and let $\int_a^b f(x)\,dx=3$. Compute $\int_a^b\int_a^x f(x)\,f(y)\,dy\,dx$. (3 marks)

Hint for (b): Consider the function $F(x)=\int_a^x f(y)\,dy$.

a)



$0 \le x \le 1$

$0 \le y \le \frac{1}{2}x$ ← 1 mark

1 mark → $\int_0^{1/2}\int_{2y}^{1} \frac{2e^{2y}}{(e^x-x)^2}\,dx\,dy = \int_0^1\int_0^{\frac{1}{2}x} \frac{2e^{2y}}{(e^x-x)^2}\,dy\,dx = \int_0^1 \left.\frac{e^{2y}}{(e^x-x)^2}\right]_0^{\frac{1}{2}x} dx$

$= \int_0^1 \frac{e^x-1}{(e^x-x)^2}\,dx = -\left.\frac{1}{e^x-x}\right]_0^1 = -\frac{1}{e-1}+1 = \boxed{\frac{e-2}{e-1}}$ ← 1 mark

b) $F'(x) = f(x)$, then

3 marks

$\int_a^b\int_a^x f(x)\,f(y)\,dy\,dx = \int_a^b f(x)\left[\int_a^x f(y)\,dy\right] dx$

$= \int_a^b F'(x)\,F(x)\,dx = \frac{1}{2}\left.[F(x)]^2\right]_a^b = \frac{1}{2}\left[[F(b)]^2-[F(a)]^2\right]$

$= \frac{1}{2}\left[\left[\int_a^b f(y)\,dy\right]^2 - \left[\int_a^a f(y)\,dy\right]^2\right] = \frac{1}{2}[9-0] = \boxed{\frac{9}{2}}$

Note (b): It can also be solved by "changing the order of integration".

9. Indicate your choice in each of the following four multiple-choice questions by a cross ("X") in the appropriate box. Each question has exactly one correct answer. A mark of zero will be given for each unanswered question, for each wrong answer, and when two or more alternatives are selected for the same question. You are not required to justify any of your answers.
Total value of this question: 8 marks (2 marks each part).

a) If $f(x,y)=e^{x^2+y^2}$ is regarded as a function of polar coordinates (r,θ) then $\dfrac{\partial f}{\partial r}=$

	0		$2r$		e^r
	e^{r^2}	X	$2re^{r^2}$		none of these

b) How many critical points has the function $f(x,y)=(1-x)(1+y^2)$?

X	0		1		2
	3		infinitely many		none of these

c) If R denotes the semidisk $x^2+y^2\le 1$, $y\ge 0$, and $f(x,y)=1$ over R, then $\iint_R f(x,y)\,dA=$

	0		1	X	$\pi/2$
	π		π^2		none of these

d) The value of $\displaystyle\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} \sin(x^3y^3)\,dx\,dy$ is

	$-\pi^3$	X	0		π
	π^3		$\arcsin(\pi^2/3)$		none of these