UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
MAT 235 Y - TEST #2

JANUARY 24, 1995

NAME: MODEL SOLUTIONS, STUDENT No.: ______
(Family name. Please PRINT.) (Given name.)

INSTRUCTIONS: Show and explain all your work in questions 1 to 8. Give your answers in the space provided. Use both sides of paper, if necessary. Do not tear out any pages. No calculators or other aids are permitted. Time allowed: 100 minutes

1. Given the surface $x^2e^z - y^3 = 8$.
   a) Find an equation of the tangent plane to the surface at the point $(-3,1,0)$. (3 marks)
   b) At what points of the given surface is the tangent plane parallel to the plane $y=0$? (3 marks)

a) $F(x,y,z) = x^2e^z - y^3 - 8$ $\qquad \nabla F(x,y,z) = (2xe^z, -3y^2, x^2e^z)$

1 mark → $\nabla F(-3,1,0) = (-6,-3,9) = \vec{n}$

1 mark → An eq. of the tangent plane at $(-3,1,0)$: $\boxed{2x+y-3z=-5}$

1 mark

b) $\nabla F(x,y,z) \parallel (0,1,0)$

1 mark

$$\left.\begin{array}{l} 2xe^z = 0 \\ -3y^2 = k \\ x^2e^z = 0 \end{array}\right\} \Rightarrow x=0,\ y=-2$$

1 mark

Answer: At any point $(0,-2,z)$

1 mark

2. Given the function $f(x,y) = x^2\sqrt{1+y}$.
   a) Compute the directional derivate $D_{\mathbf{u}}f(-2,3)$. Where $u=(2,-1)$. (3 marks)
   b) Find the unit vector v that minimizes the value of $D_{\mathbf{v}}f(-2,3)$. (2 marks)

a) $\nabla f(x,y) = \left(2x\sqrt{1+y}, \dfrac{x^2}{2\sqrt{1+y}}\right)$ $\qquad \nabla f(-2,3) = (-8,1)$ ← 1 mark

1 mark

$D_{\vec{u}}f(-2,3) = \frac{1}{\sqrt{5}}(2,-1)\cdot(-8,1) = \boxed{-\frac{17}{\sqrt{5}} \quad \left(\text{or } -\frac{17\sqrt{5}}{5}\right)}$

1 mark →

b) $\vec{v} = -\dfrac{1}{\|\nabla f(-2,3)\|}\nabla f(-2,3) = \boxed{-\frac{1}{\sqrt{65}}(-8,1) \quad \left(\text{or } \frac{\sqrt{65}}{65}(8,-1)\right)}$

2 marks

3. Let $f(x,y)=y-x^5e^y$. Assume that x and y are defined implicitly in terms of t by the equations $t^5 - tx^3 = 2$, and $t^3 + ty - 2y = 1$. Compute $\frac{df}{dt}$ at $t=1$. (5 marks)

1 mark

mark $\rightarrow$ $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}; \qquad \frac{\partial f}{\partial x} = -5x^4e^y \qquad \frac{\partial f}{\partial y} = 1 - x^5e^y$$

1 mark $\rightarrow$ $$\frac{dx}{dt} = -\frac{5t^4 - x^3}{-3tx^2} = \frac{5t^4-x^3}{3tx^2} \qquad \frac{dy}{dt} = -\frac{3t^2+y}{t-2} = \frac{3t^2+y}{2-t}$$

mark $\rightarrow$ at $t=1$: $x=-1$, and $y=0$

$$\left.\frac{dx}{dt}\right|_{t=1} = 2 \qquad \left.\frac{dy}{dt}\right|_{t=1} = 3$$

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=-1\\y=0}} = -5 \qquad \left.\frac{\partial f}{\partial y}\right|_{\substack{x=-1\\y=0}} = 2$$

1 mark $\rightarrow$ $$\left.\frac{df}{dt}\right|_{t=1} = (-5)(2) + (2)(3) = \boxed{-4}$$

4. Given the function $f(x,y) = \sin(x-y) + \cos(x+y)$.
   a) Find all critical points of this function on the region $0 < x < \pi$, $0 < y < \pi$. (3 marks)
   b) Use the second derivative test to classify the critical points found in part (a). (3 marks)
   c) Find the extreme values of the function f on the region $0 \le x \le \pi$, $0 \le y \le \pi$. (3 marks)

a) $\frac{\partial f}{\partial x} = \cos(x-y) - \sin(x+y)$ $\qquad \frac{\partial f}{\partial y} = -\cos(x-y) - \sin(x+y)$

1 mark

For $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$: $\quad \sin(x+y) = 0 = \cos(x-y)$

2 marks

Then: $\left.\begin{array}{l} x+y = \pi \\ x-y = \frac{\pi}{2} \end{array}\right\}$ $\boxed{x = \frac{3\pi}{4} \qquad y = \frac{\pi}{4}}$

b) $\frac{\partial^2 f}{\partial x^2} = -\sin(x-y) - \cos(x+y)$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \sin(x-y) - \cos(x+y)$

$\frac{\partial^2 f}{\partial y^2} = -\sin(x-y) - \cos(x+y)$

1 mark

1 mark

$D = \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} = -4 < 0$

1 mark

$\boxed{\text{Saddle pt at } \left(\frac{3\pi}{4}, \frac{\pi}{4}\right)}$

c) When $x = 0$: $f(0,y) = \sin(-y) + \cos y = \cos y - \sin y$

When $y = 0$: $f(x,0) = \sin x + \cos x$

When $x = \pi$: $f(\pi, y) = \sin(\pi - y) + \cos(\pi + y)$
$= \sin y - \cos y$

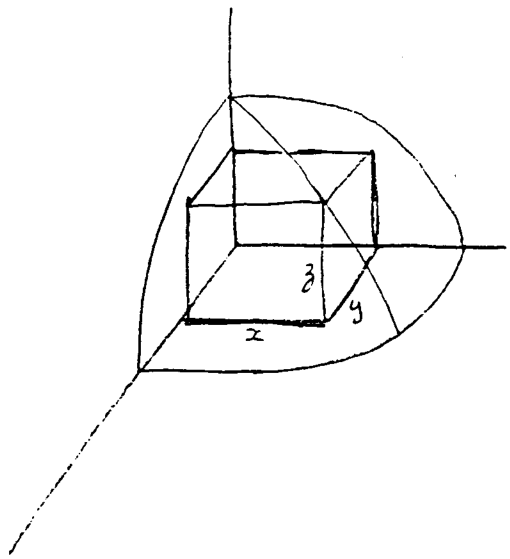
When $y = \pi$: $f(x,\pi) = \sin(x-\pi) + \cos(x+\pi)$
$= -\sin x - \cos x$

omit

?(1+3)

$\boxed{\begin{array}{l} \text{Abs max: } \sqrt{2}, \text{ at } \left(\frac{\pi}{4}, 0\right) \text{ and at } \left(\pi, \frac{3\pi}{4}\right) \\ \text{Abs min: } -\sqrt{2}, \text{ at } \left(0, \frac{3\pi}{4}\right) \text{ and at } \left(\frac{\pi}{4}, \pi\right) \end{array}}$

5. A rectangular box is placed inside the ellipsoid $\frac{x^2}{12} + \frac{y^2}{27} + \frac{z^2}{3} = 1$ with sides parallel to the axes. What dimensions will give the box with the maximum possible volume? (7 marks)



2 marks

Maximize: $V = 8xyz$

Subject to: $\frac{x^2}{12} + \frac{y^2}{27} + \frac{z^2}{3} = 1$

$\nabla V = (8yz,\ 8xz,\ 8xy)$

$\nabla g = \left(\frac{2x}{12},\ \frac{2y}{27},\ \frac{2z}{3}\right)$

2 marks

$8yz \overset{1}{=} \frac{2\lambda x}{12}, \quad 8xz \overset{2}{=} \frac{2\lambda y}{27}, \quad 8xy \overset{3}{=} \frac{2\lambda z}{3}$

$\frac{x^2}{12} + \frac{y^2}{27} + \frac{z^2}{3} \overset{4}{=} 1$

2 marks

From 1, 2, 3: $24xyz = \lambda\left[\frac{2x^2}{12} + \frac{2y^2}{27} + \frac{2z^2}{3}\right] = 2\lambda$

Then $xyz = \frac{\lambda}{12}$. From 1: $x = 2$

From 2: $y = 3$

From 3: $z = 1$

1 mark → Optimal dimensions: $\boxed{4 \times 6 \times 2}$