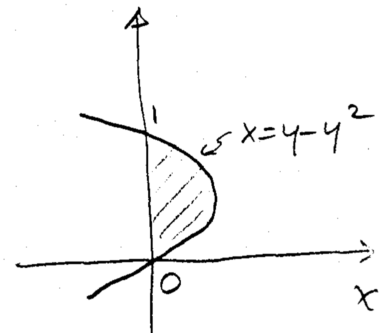
a) (7 marks) Use Green's Theorem to evaluate $\int_C (x^2 - y)dx + xydy$, where the curve C is the boundary the region enclosed between $x = y - y^2$ and $x=0$, traversed once in the counterclockwise direction .

b) (8 marks) Use the Divergence Theorem to evaluate $\iint_S (3x^2y^2 - y^4 + 5z^2)dS$, where S is the sphere $x^2 + y^2 + z^2 = 1$.



$$\begin{cases} 0 \le y \le 1 \\ 0 \le x \le y-y^2 \end{cases}$$

a) $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = y+1$

$$\int_C (x^2-y)dx + xy\,dy = \iint_R (y+1)\,dA$$

$$= \int_0^1 \int_0^{y-y^2} (y+1)\,dx\,dy$$

$$= \int_0^1 (y+1)(y-y^2)\,dy = \int_0^1 (y-y^3)\,dy = \frac{1}{2}y^2 - \frac{1}{4}y^4 \Big]_0^1 = \boxed{\frac{1}{4}}$$

b) $\vec{n} = (x, y, z)$; $F \cdot \vec{n} = 3x^2y^2 - y^4 + 5z^2$

Take $F = (3xy^2, -y^3, 5z)$, then

$$\nabla \cdot F = 3y^2 - 3y^2 + 5 = 5$$

$$\iint_S (3x^2y^2 - y^4 + 5z^2)\,ds = \iiint_R 5\,dV = 5\left[\frac{4}{3}\pi\right] = \boxed{\frac{20\pi}{3}}$$

(5 marks) Verify Stokes' Theorem for the surface S given by $x^2+3y^2+z^2=7$, $z \geq 2$ and the vector field $F(x,y,z)=(x,5x,yz)$ by explicitly evaluating both the relevant line and surface integrals.

Computing $\int_C F \cdot dr$

C: $x^2+3y^2=3$; $z=2$

$$\begin{cases} x=\sqrt{3}\cos\theta \\ y=\sin\theta \\ z=2 \end{cases} \quad 0 \leq \theta \leq 2\pi$$

$$\int_C F \cdot dr = \int_0^{2\pi} (\sqrt{3}\cos\theta, 5\sqrt{3}\cos\theta, 2\sin\theta)\cdot(-\sqrt{3}\sin\theta, \cos\theta, 0)\,d\theta$$

$$= \int_0^{2\pi} (-3\sin\theta\cos\theta + 5\sqrt{3}\cos^2\theta)\,d\theta$$

$$= 5\sqrt{3}\left[\tfrac{1}{2}\theta\right]_0^{2\pi} = \boxed{5\pi\sqrt{3}}$$

7

Computing $\iint_S (\text{Curl}\, F \cdot \vec{n})\, dS$

8

$$\text{Curl}\, F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & 5x & yz \end{vmatrix} = (z, 0, 5)$$

$$\vec{n} = \frac{(2x, 6y, 2z)}{\sqrt{4x^2+36y^2+4z^2}} = \frac{1}{\sqrt{x^2+9y^2+z^2}}(x, 3y, z)$$

$$\text{Curl}\, F \cdot \vec{n} = xz + 5z$$

$$dS = \sqrt{1+\left(-\frac{x}{z}\right)^2+\left(-\frac{3y}{z}\right)^2}\, dA = \frac{\sqrt{x^2+9y^2+z^2}}{z}\, dA$$

$$\iint_S (\text{Curl}\, F \cdot \vec{n})\, dS = \iint_R (x+5)\, dA = \iint_R 5\, dA$$

$$= 5\left(\text{Area of ellipse } \frac{x^2}{3}+y^2=1\right) = 5\pi(\sqrt{3})(1) = \boxed{5\pi\sqrt{3}}.$$

(10 marks) Find the function $f(t)$ that satisfies the conditions:
$f''' - 3f' - 2f = 0$, $f(0) = -1$, $f'(0) = 5$ and $f''(0) = 0$.

$r^3 - 3r - 2 = 0$; $(r+1)(r^2 - r - 2) = 0$

$(r+1)(r+1)(r-2) = 0$

$r_1 = -1$, $r_2 = -1$; $r_3 = 2$

General soln.: $f(t) = c_1 e^{-t} + t c_2 e^{-t} + c_3 e^{2t}$

Then: $f'(t) = -c_1 e^{-t} + c_2 e^{-t} - t c_2 e^{-t} + 2 c_3 e^{2t}$

$f''(t) = c_1 e^{-t} - c_2 e^{-t} - c_2 e^{-t} + t c_2 e^{-t} + 4 c_3 e^{2t}$

Then: $\begin{cases} c_1 + c_3 = -1 \\ -c_1 + c_2 + 2c_3 = 5 \\ c_1 - 2c_2 + 4c_3 = 0 \end{cases}$; $\left.\begin{array}{l} c_1 + c_3 = -1 \\ -c_1 + 8c_3 = 10 \end{array}\right\}$ $c_3 = 1$

$c_1 = -2$ $c_2 = 1$

$$\boxed{f(t) = -2e^{-t} + te^{-t} + e^{2t}}$$