PLEASE HAND IN

UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

APRIL/MAY EXAMINATIONS 1997

MAT 235 Y – CALCULUS FOR PHYSICAL AND LIFE SCIENCE II

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INSTRUCTIONS:

1. ATTEMPT ALL QUESTIONS.
2. SHOW AND EXPLAIN YOUR WORK IN ALL QUESTIONS.
3. GIVE YOUR ANSWERS IN THE SPACE PROVIDED. USE BOTH SIDES OF PAPER, IF NECESSARY.
4. DO NOT TEAR OUT ANY PAGES.
5. NO CALCULATORS OR ANY OTHER AIDS ARE PERMITTED.
6. THIS EXAM CONSISTS OF EIGHT QUESTIONS. THE VALUE OF EACH QUESTION IS INDICATED (IN BRACKETS) BY THE QUESTION NUMBER.
7. THIS EXAM IS WORTH 40% OF YOUR FINAL GRADE.
8. TIME ALLOWED: 3 HOURS.
9. PLEASE WRITE YOUR NAME, YOUR STUDENT NUMBER, AND YOUR SIGNATURE IN THE SPACE PROVIDED AT THE BOTTOM OF THIS PAGE.

PLEASE DO NOT WRITE HERE

QUESTION NUMBER	QUESTION VALUE	GRADE
1	10	
2	15	
3	15	
4	10	
5	10	
6	15	
7	15	
8	10	
TOTAL:	100	

NAME: MODEL SOLUTIONS

(FAMILY NAME. PLEASE PRINT.) (GIVEN NAME.)

Consider the curve $r(t) = (t^3 - 2t, 3t^2, 2t\sqrt{3})$.

a) (5 marks) Compute its arclength from $t = 1$ to $t = 3$.

b) (5 marks) Compute its curvature at $t = 0$.

5

a) $\vec{r}'(t) = (3t^2 - 2, 6t, 2\sqrt{3})$

$\|\vec{r}'(t)\| = \sqrt{(3t^2-2)^2 + (6t)^2 + (2\sqrt{3})^2}$

$= \sqrt{9t^4 - 12t^2 + 4 + 36t^2 + 12}$ 3

$= \sqrt{9t^4 + 24t^2 + 16} = \sqrt{(3t^2+4)^2}$

$\|\vec{r}'(t)\| = 3t^2 + 4$

$l = \int_1^3 (3t^2+4)\,dt = t^3 + 4t\Big]_1^3 = \boxed{34}$ 2

5

b) $\vec{r}''(t) = (6t, 6, 0)$ 1

$\vec{r}'(0) = (-2, 0, 2\sqrt{3})$ $\quad \vec{r}''(0) = (0, 6, 0)$ 1

$\vec{r}'(0) \times \vec{r}''(0) = (-12\sqrt{3}, 0, -12)$ 1

$\|\vec{r}'(0)\| = 4, \quad \|\vec{r}''(0)\| = 24$ 1

$\kappa = \frac{24}{4^3} = \boxed{\frac{3}{8}}$ 1

For Felix Recio

(5 marks) Find the absolute maximum and the absolute minimum of the function $f(x,y) = x^2 + xy + y^2 + 3y$ on the disk $x^2 + y^2 \leq 9$. Indicate the coordinates of the points where these extremes are reached.

Finding the critical pts: 5

$f_x = 2x + y \qquad f_y = x + 2y + 3$

For $f_x = 0 = f_y$: $\begin{cases} 2x + y = 0 \\ x + 2y = -3 \end{cases} \qquad x = 1 \ ; \ y = -2$

On the boundary: $(x^2 + y^2 = 9)$ 7

$\begin{cases} 2x + y = 2\lambda x \\ x + 2y + 3 = 2\lambda y \end{cases} \qquad \dfrac{2x+y}{x} = \dfrac{x + 2y + 3}{y} \qquad y^2 = x^2 + 3x$

Then: $x^2 + x^2 + 3x = 9 \qquad 2x^2 + 3x - 9 = 0$

$(2x - 3)(x + 3) = 0 \qquad x = -3 \quad$ or $\quad x = \frac{3}{2}$

$x = -3 \ ; \ y = 0 \qquad x = \frac{3}{2} \ ; \ y = \pm \frac{3\sqrt{3}}{2}$

Testing the four possible extrema: 3

Abs. minimum: -3 at $x = 1 \ ; \ y = -2$

Abs. maximum: $9 + \frac{27\sqrt{3}}{4}$ at $x = \frac{3}{2}, \ y = \frac{3\sqrt{3}}{2}$

$9 + \cancel{y\sqrt{9-y^2}} + 3\sqrt{9-y^2}$

0 marks) **Evaluate the integral** $\iiint_R \frac{8y}{4+(x^2+y^2+z^2)^2}\,dV$, where R is the region $0 \le z \le \sqrt{2-x^2-y^2}$, $y \ge |x|$.

Using "Spherical coord.":

$$\iiint_R \frac{8y}{4+(x^2+y^2+z^2)^2}\,dV$$

3

$$= 16\int_0^{\sqrt{2}}\int_{\pi/4}^{\pi/2}\int_0^{\pi/2} \frac{\rho^3 \sin\theta \sin^2\phi}{4+\rho^4}\,d\phi\,d\theta\,d\rho$$

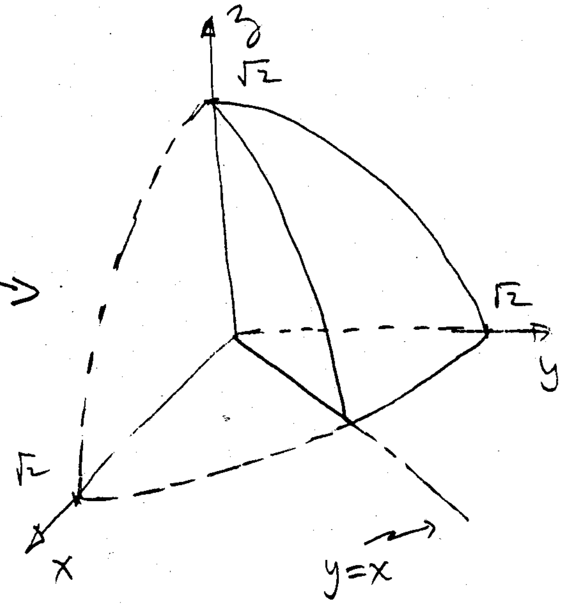
1 2

$$= 16\left[-\cos\theta\right]_{\pi/4}^{\pi/2}\left[\tfrac{1}{2}\phi\right]_0^{\pi/2}\left[\tfrac{1}{4}\ln(4+\rho^4)\right]_0^{\sqrt{2}}$$

1 1 1

$$= 16\left[\frac{\sqrt{2}}{2}\right]\left[\frac{\pi}{4}\right]\left[\frac{1}{4}\ln 2\right]$$

$$= \boxed{\frac{\pi\sqrt{2}\ln 2}{2}}$$



$$\begin{cases} 0 \le \rho \le \sqrt{2} \\ \frac{\pi}{4} \le \theta \le \frac{\pi}{2} \\ 0 \le \phi \le \frac{\pi}{2} \end{cases}$$ 1

[redacted]

(10 marks) **Determine the coordinates of the centroid of the region $2x^2+2y^2 \le z \le 3-x^2-y^2$. (Assume that the density is constant.)**

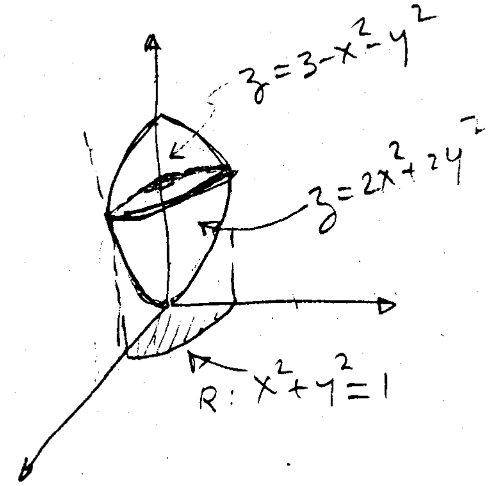
Use "Cylindrical coord."

$$V=\int_0^{2\pi}\int_0^1\int_{2r^2}^{3-r^2} r\,dz\,dr\,d\theta$$

3

$$= 2\pi\int_0^1 (3r-3r^3)\,dr$$

$$= 2\pi\left[\frac{3r^2}{2}-\frac{3r^4}{4}\right]_0^1 = \frac{3\pi}{2}$$



$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$2r^2 \le z \le 3-r^2$

$$M_{xy}=\int_0^{2\pi}\int_0^1\int_{2r^2}^{3-r^2} rz\,dz\,dr\,d\theta$$

$$= 2\pi\int_0^1 r\left[\tfrac{1}{2}z^2\right]_{2r^2}^{3-r^2} dr$$

4

$$= \pi\int_0^1 r(9-6r^2-3r^4)\,dr$$

$$= \pi\left[\frac{9r^2}{2}-\frac{3r^4}{2}-\frac{r^6}{2}\right]_0^1 = \frac{5\pi}{2}$$

3

$$\bar{z} = \frac{\frac{5\pi}{2}}{\frac{3\pi}{2}} = \frac{5}{3}$$

~~Because~~ of the symmetries: $\boxed{\bar{x}=\bar{y}=0;\ \bar{z}=\frac{5}{3}}$