PG

8. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y,z) = 2z\mathbf{i} + 4x\mathbf{j} + 5y\mathbf{k}$$

and C is the curve of intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$ and is traversed in the counterclockwise direction as viewed down from the positive z-axis.

Parameterize C via:

$$\underline{r}(t) = (2\cos t,\ 2\sin t,\ 4 + 2\cos t) \qquad (0 \le t < 2\pi)$$

$$\underline{r}'(t) = (-2\sin t,\ 2\cos t,\ -2\sin t)$$

Then

$$\int_C \underline{F} \cdot d\underline{r} = \int_0^{2\pi} (8 + 4\cos t,\ 8\cos t,\ 10\sin t) \cdot (-2\sin t,\ 2\cos t,\ -2\sin t)\,dt$$

$$= \int_0^{2\pi} -16\sin t - 8\cos t\sin t + 16\cos^2 t - 20\sin^2 t$$

$$= \underbrace{16\cos t \Big|_0^{2\pi}}_{0} - \underbrace{4\int_0^{2\pi} \sin 2t\,dt}_{0}$$

$$+ \int_0^{2\pi} 16 - 16\sin^2 t - 20\sin^2 t\,dt$$

$$= 16t\Big|_0^{2\pi} - 36\int_0^{2\pi} \sin^2 t\,dt$$

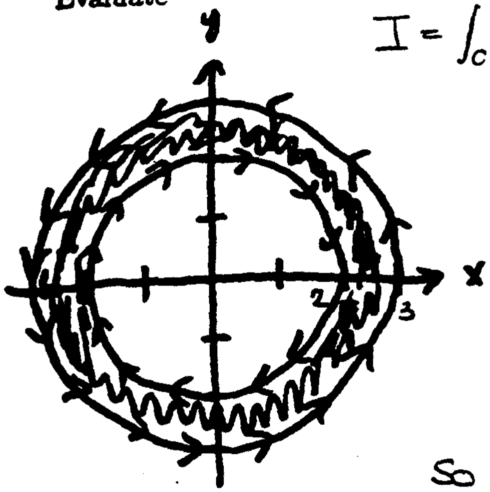
$$= 32\pi - \left[36\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Big|_0^{2\pi}\right]$$

$$= 32\pi - 36\pi$$

$$= -4\pi. \quad //$$

9. Let R be the region consisting of all points (x, y) satisfying $4 \le x^2 + y^2 \le 9$. Let C be its boundary curve, oriented as in the drawing (so C consists of two circles). Evaluate

$$I = \int_C (-x^3y^2)\,dx + (x^2y^3)\,dy.$$



As C is oriented positively we can use Green's theorem with

$$P = -x^3y^2 \quad Q = x^2y^3.$$

Then $Q_x = 2xy^3 \quad P_y = -2yx^3$.

So Green's theorem gives:

$$I = \int_C P\,dx + Q\,dy$$

$$= \iint_R (Q_x - P_y)\,dx\,dy$$

$$= 2 \iint_D xy(y^2+x^2)\,dx\,dy$$

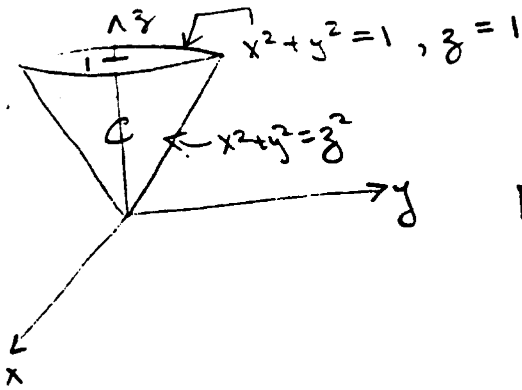
set $x = r\cos\theta$, $y = r\sin\theta$, $2 \le r \le 3$, $0 < \theta < 2\pi$

$$= 2 \int_0^{2\pi}\int_2^3 r^5 \cos\theta \sin\theta \, dr\, d\theta$$

$$= \underbrace{\int_0^{2\pi} \sin 2\theta \, d\theta}_{=0} \cdot \int_2^3 r^5\,dr$$

$$= 0 \quad //$$

10. Find the centroid $(\bar{x}, \bar{y}, \bar{z})$ of the portion of the cone $x^2 + y^2 = z^2$ above the plane $z = 0$ and beneath the plane $z = 1$.



By symmetry, $\bar{x} = \bar{y} = 0$.

$$M_{xy} = \iiint_C z \, dz \, dx \, dy$$

$$= \int_0^{2\pi} \int_0^1 \int_r^1 z \, dz \; r \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left. \frac{z^2}{2} \right|_r^1 r \, dr$$

$$= \pi \int_0^1 (1 - r^2) \, r \, dr$$

$$= \pi \left(\frac{r^2}{2} - \frac{r^4}{4} \right) \Big|_0^1$$

$$= \frac{\pi}{4}.$$

Also $M = \iiint_C dz \, dx \, dy = \int_0^{2\pi} \int_0^1 \int_r^1 r \, dz \, dr \, d\theta$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 r - r^2 \, dr$$

$$= 2\pi \cdot \left(\frac{r^2}{2} - \frac{r^3}{3} \right) \Big|_0^1$$

$$= 2\pi \cdot \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$= \pi/3.$$

So $\bar{z} = \dfrac{M_{xy}}{M} = \dfrac{\pi/4}{\pi/3} = \dfrac{3}{4}$.

So the centroid is $(0, 0, 3/4)$. //

11. Find the flux of the vector field $\mathbf{v}(x, y, z) = x^2\mathbf{i} + yz\mathbf{j} + z\mathbf{k}$ out of the surface of the cube $0 \le x, y, z \le 1$ (either directly or by the divergence theorem, your choice).

Use the divergence theorem:

$$\text{flux} = \int_S \underline{F} \cdot d\underline{S} = \int_0^1 \int_0^1 \int_0^1 (\operatorname{div} F)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 \int_0^1 (2x + z + 1)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 x^2 + (z+1)x \Big|_0^1 \, dy\, dz$$

$$= \int_0^1 \int_0^1 (1 + z + 1)\, dy\, dz$$

$$= \int_0^1 z + 2\, dz \cdot \int_0^1 dy$$

$$= z^2/2 + 2z \Big|_0^1$$

$$= \frac{1}{2} + 2$$

$$= \frac{5}{2} \quad //$$

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12. Evaluate

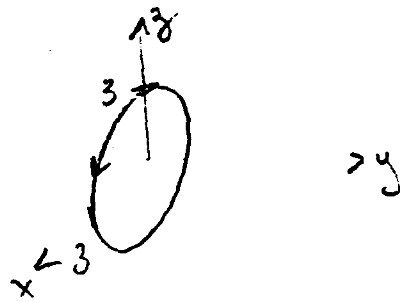
$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS$$

where S is the surface

$$\begin{aligned} x^2 + y^2 + z^2 &= 9 \\ y &\geq 0 \end{aligned}$$

with normals having positive y-coordinate, and where $\mathbf{F} = (2y, x^3, z + x)$.

Use Stoke's theorem: the boundary ∂S of S is the circle $x^2 + z^2 = 9$, $y = 0$ oriented as pictured,



Then $\displaystyle\iint_S (\nabla \times \underline{F}) \cdot \underline{n}\, dS$

$$= \iint_S (\nabla \times \underline{F}) \, d\underline{S}$$

$$= \int_{\partial S} \underline{F} \cdot d\underline{s}$$

$\underline{s}(t) = (3\cos t, 0, -3\sin t)$

$\underline{s}'(t) = (-3\sin t, 0, -3\cos t)$

$0 < t \leq 2\pi$

$$\stackrel{\downarrow}{=} \int_0^{2\pi} (0, -27\sin^3 t, 3(\cos t - \sin t)) \cdot (-3\sin t, 0, -3\cos t)\, dt$$

$$= \int_0^{2\pi} -9 (\cos^2 t - \cos t \sin t)\, dt$$

$$= -9\left(\int_0^{2\pi} \cos^2 t\, dt - \frac{1}{2}\underbrace{\int_0^{2\pi} \sin 2t\, dt}_{=0}\right.$$

$$= -9\left(\frac{t}{2} + \frac{\sin 2t}{4}\right)\Big|_0^{2\pi}$$

$$= -9\pi. \quad //$$