APRIL/MAY
EXAM 199
PG1

1. Find the minimum value and the maximum value obtained by the function

$$f(x,y) = xy$$

on the disk $x^2 + y^2 \le 1$.

Find critical points of f in open disk $x^2+y^2<1$.

$f_x = y$ $\quad f_y = x$ $\quad$ For a critical pt, $f_x = f_y = 0$

ie. $x = y = 0$. $\quad f_{xy} = 1 \quad f_{xx} = 0 \quad f_{yy} = 0$

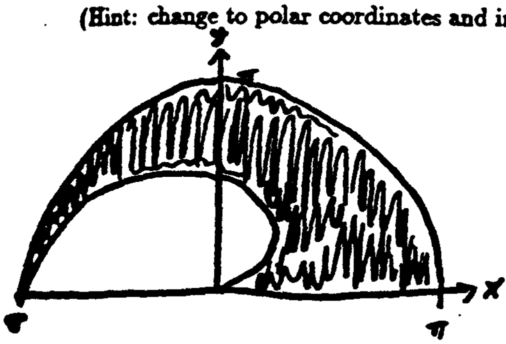
$D(0,0) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$ so $(0,0)$, the only critical pt, is a saddle point (hence not a max or min). So max/min occurs on the unit circle. Parameterize this circle by $x = \cos\theta$, $y = \sin\theta$ $(0 \le \theta < 2\pi)$. Then $f(x,y) = \cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$

It is thus clear that the maximum value of f is $1/2$ (attained at $\theta = \pi/4$, ie. $(1/\sqrt{2}, 1/\sqrt{2})$) and the minimum value of f is $-1/2$ (attained at $\theta = 3\pi/4$, ie $(-1/\sqrt{2}, 1/\sqrt{2})$). //

4. Consider the region R drawn below, its boundary curves being the positive x axis, the circle $x^2 + y^2 = \pi^2$, and the spiral $r = \theta$. Compute

$$I = \iint_R e^{(x^2+y^2)^{3/2}}\,dx\,dy.$$

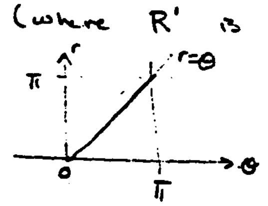
(Hint: change to polar coordinates and integrate with respect to θ *first.*)



$$I = \iint_R e^{(x^2+y^2)^{3/2}}\,dx\,dy$$

$$= \iint_{R'} e^{r^3}\, r\, d\theta\, dr$$

(where R' is region



)

$$= \int_0^\pi r e^{r^3}\left(\int_0^r 1\, d\theta\right) dr$$

$$= \int_0^\pi r^2 e^{r^3}\, dr$$

$$= \frac{1}{3} e^{r^3}\Big|_{r=0}^{r=\pi}$$

$$= \frac{1}{3}\left(e^{\pi^3} - 1\right). \quad //$$

2. Consider the function

$$f(x,y) = 2x^3 + 3x^2 + y^3 - 3y$$

defined on the whole x-y-plane.

a. Find all the critical points of f.

b. For each critical point, determine whether it is a local minimum, a local maximum, or a saddle.

(a) $\left.\begin{array}{l} f_x = 6x^2 + 6x \\ f_y = 3y^2 - 3 \end{array}\right\}$ for c.p. $f_x = f_y = 0$

But $f_x = 0 \iff 6x(x+1) = 0$
$\iff x = 0$ or $x = -1$

and $f_y = 0 \iff y = 1$ or -1

So the critical points of f are: $(0,1)$, $(0,-1)$, $(-1,1)$, $(-1,-1)$.

(b) $\left.\begin{array}{l} f_{xx} = 12x + 6 \\ f_{xy} = 0 = f_{yx} \\ f_{yy} = 6y \end{array}\right\} \Rightarrow D(x,y) = \begin{vmatrix} 12x+6 & 0 \\ 0 & 6y \end{vmatrix} = (12x+6)(6y)$

Now $D(0,1) = (6)(6) = 36 > 0$ and $f_{yy} = 6 > 0$ so $(0,1)$ is a local max.

$D(0,-1) = (6)(-6) < 0$ so $(0,-1)$ is a saddle point.

$D(-1,1) = (-6)(6) < 0$ so $(-1,1)$ is a saddle point.

$D(-1,-1) = (-6)(-6) > 0$ and $f_{yy} = -6 < 0$ and $f_{xx} = -6 < 0$ so $(-1,-1)$ is a local min

3. Find the equation

$$Ax + By + Cz = 1$$

of the plane that passes through the point $(x, y, z) = (1, 2, 3)$ such that the tetrahedron that the plane cuts off in the first octant $x, y, z \geq 0$ has the smallest volume. (Note that the variables here are A, B, and C and they are required to be positive!)

Let $V(A,B,C)$ be the volume of the tetrahedron:

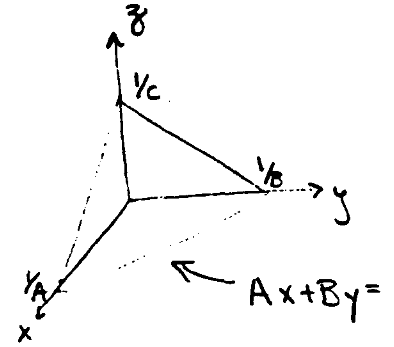
$$V(A,B,C) = \int_0^{1/B} \int_0^{\frac{1-By}{A}} \frac{1-Ax-By}{C}\, dx\, dy$$

$$= \frac{1}{C} \int_0^{1/B} (1-By)x - \tfrac{A}{2}x^2 \Big|_0^{\frac{1-By}{A}} dy$$

$$= \frac{1}{2CA} \int_0^{1/B} (1-By)^2\, dy$$

$$= \frac{-1}{2ABC} \frac{1}{3} (1-By)^3 \Big|_0^{1/B}$$

$$= \frac{1}{6ABC}.$$



We want to minimize $V(A,B,C)$ subject to $g(A,B,C) = A+2B+3C = 0$.

and $A, B, C > 0$.

Set
$$\left.\begin{array}{l} V_A = \lambda g_A \\ V_B = \lambda g_B \\ V_C = \lambda g_C \\ g = 0 \end{array}\right\} \iff \begin{array}{l} \frac{-1}{6BCA^2} = \lambda \\ \frac{-1}{6B^2CA} = 2\lambda \\ \frac{-1}{6ABC^2} = 3\lambda \\ g = 0 \end{array}$$

$$\Longrightarrow \frac{-2}{6BCA^2} = \frac{-1}{6B^2CA} \Rightarrow 2B =$$

$$\Longrightarrow \frac{2}{3} = \frac{6ABC^2}{6B^2CA} \Rightarrow 2B = 3C$$

So $A = 2B = 3C$ for critical points. Substituting in $g=0$ gives $C = 1/9$, $B = 1/6$, $A = 1/3$. Geometrically, it is clear that this gives a (the) minimum value for V subject to the constraint. So the plane desired is $\frac{1}{3}x + \frac{1}{6}y + \frac{1}{9}z =$

6. Find the surface area of the part of the saddle $z = xy + 100$ which lies in the solid cylinder $x^2 + y^2 \le 9$.

Let R be the region $x^2+y^2 \le 9$.

Then $S.A. = \iint_R \sqrt{1+f_x^2+f_y^2}\,dx\,dy$

$$= \iint_R \sqrt{1+y^2+x^2}\,dx\,dy$$

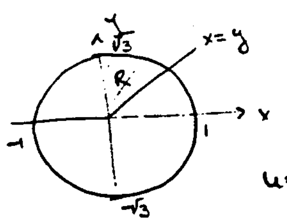
Set $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ 0 \le r \le 3 \\ 0 \le \theta < 2\pi \end{array}\right\}$ $\longrightarrow$ $\overset{\downarrow}{=} \int_0^{2\pi}\int_0^3 r\sqrt{1+r^2}\,dr\,d\theta$

$$= 2\pi \cdot \frac{1}{3}(1+r^2)^{3/2}\Big|_0^3$$

$$= \frac{2\pi}{3}\left(10^{3/2} - 1\right) /\!/$$

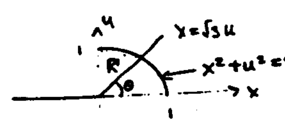
7. Find the area of the region given by

$$x^2 + \frac{y^2}{3} \le 1$$
$$x \ge y$$
$$y \ge 0.$$



$A = \iint_R dx\,dy$

let $u = y/\sqrt{3}$, $du = dy/\sqrt{3}$ $\longrightarrow$ $\overset{\downarrow}{=} \iint_{R'} \sqrt{3}\,dx\,du$ where R' is given by



set $x = r\cos\theta$, $y = r\sin\theta$, $0 \le r \le 1$, $0 \le \theta < 2\pi$ $\longrightarrow$ $\overset{\downarrow}{=} \sqrt{3}\int_{\pi/3}^{\pi/2}\int_0^1 r\,dr\,d\theta$ note $\theta = \pi/3$

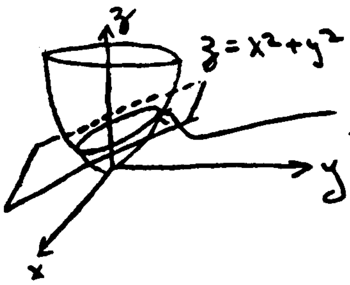
$$= (\sqrt{3})\left(\frac{\pi}{2} - \frac{\pi}{3}\right)\left(\frac{r^2}{2}\Big|_0^1\right)$$

$$= \frac{\sqrt{3}\,\pi}{12}. /\!/$$

6. **Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$.**

P1

The two surfaces meet here because their intersection is where $\begin{cases} z = x^2+y^2 \\ z = 2y \end{cases}$



$\begin{cases} x^2+(y-1)^2=1 \\ z=2y \end{cases}$

ie. $\begin{cases} 2y = x^2+y^2 \\ z=2y \end{cases}$

ie. $\begin{cases} x^2 + (y^2-2y+1) - 1 = 0 \\ z=2y \end{cases}$

So the projection of the region they bound onto the xy-plane is $x^2+(y-1)^2 \le 1$. Let R be the disk $x^2+(y-1)^2 \le 1$.

Then $$V = \iint_R \left[2y - (x^2+y^2)\right] dx\,dy$$

$$= -\iint_R \left[x^2+(y-1)^2 - 1\right] dx\,dy$$

$$= -\iint_{R'} (x^2+u^2-1)\, dx\,du$$

R' = unit disk in xu-plane where $u = y-1$, $du = dy$

set $x = r\cos\theta$, $u = r\sin\theta$, $r \le 1$, $0 \le \theta < 2\pi$

$$\stackrel{\uparrow}{=} -\int_0^{2\pi}\int_0^1 (r^2-1)\, r\, dr\, d\theta$$

$$= -2\pi \cdot \left(\frac{r^4}{4} - \frac{r^2}{2}\right)\Big|_0^1$$

$$= -2\pi \cdot (1/4 - 1/2)$$

$$= \pi/2.$$

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