UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS

MAT 235 Y - CALCULUS FOR PHYSICAL AND LIFE SCIENCE II

TEST #1. OCTOBER 21, 1997

NAME: MODEL SOLUTIONS
(Family name. Please PRINT.), **(Given name.)** **STUDENT No.:** ____________

INSTRUCTIONS: This test consists of six questions. The value of each question is indicated (in brackets) by the question number. Total marks: 100.
Show all your work in all questions. Give your answers in the space provided. Use both sides of paper, if necessary. Do not tear out any pages.
No calculators or any other aids are permitted. Duration: 2 hours.
This test is worth 20% of your course grade.

1. a) (5 marks) Find the angle between the vectors **u** = (1 , 0 , - 1) and v = (4 , - 1 , 1).
   b) (5 marks) Find the area of the parallelogram whose vertices are the points (1 , - 1 , 2), (2 , - 1 , 3), (0 , 1 , 2), and (1 , 1 , 3).

a) Let θ denote the angle between the vectors $\vec{u}$ and $\vec{v}$, then

1 mark

2 marks $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \dfrac{3}{\sqrt{2}\,\sqrt{18}} = \dfrac{1}{2}$ $\boxed{\theta = \frac{\pi}{3}}$ 1 mark

1 mark

b) Let $A = (1,-1,2)$, $B = (2,-1,3)$, $C = (0,1,2)$, and $D = (1,1,3)$.

Then $\vec{AB} = (1,0,1)$ and $\vec{AC} = (-1,2,0)$ 1 mark

1 mark

$\vec{AB}\times\vec{AC} = (-2,-1,2)$, and $\|\vec{AB}\times\vec{AC}\| = 3$

2 marks

$\boxed{\text{The area of the parallelogram is } 3}$

1 mark

2. Given the point $A = (0, 3, -1)$, the line L with parametric equations $(x, y, z) = (2 - t, 1 + 2t, 3t)$, and the plane α with equation $2x - y + z = 5$.

a) (10 marks) Find parametric equations of the straight line that passes through the point A and through the point of intersection of L and α.

b) (10 marks) Find an equation of the plane that contains both, the point A and the line L.

c) (10 marks) Find the coordinates of the point of α which is closest to the point A.

a) At the point of intersection of L and α: 3 marks
$2(2-t)-(1+2t)+3t=5$, then $t=-2$. The coordinates of the point of intersection I are: $I=(4,-3,-6)$ 3 marks
Param. eq. of the line through A and I are:
$\boxed{(x,y,z) = (4\lambda, 3-6\lambda, -1-5\lambda)}$ 4 marks

b) $B=(2,1,0)$ is a point on L and $\vec{v}=(-1,2,3)$ is a vector director of L, then $\vec{AB}=(2,-2,1)$ 3 marks and
3 marks $\vec{AB}\times\vec{v} = (8,7,-2)$. An equation of the plane that 4 marks contains both, A and L is: $\boxed{8x+7y-2z=23}$

c) Param. eq. of the line through A and perpendicular to α are: $(x,y,z)=(2t, 3-t, -1+t)$. 4 marks
At the intersection point: 3 marks
$4t-(3-t)-1+t=5$, then $t=\frac{3}{2}$.
The coordinates of the point of α closest to A are:
$\boxed{\left(3, \frac{3}{2}, \frac{1}{2}\right)}$ 3 marks

3. a) (10 marks) Compute the arclength of the curve $r(t) = (3t^2, 3t - t^3)$ between the points corresponding to $t = 1$ and $t = 2$.

b) (10 marks) Compute the curvature of the curve $r(t) = (\sin t, t - \cos t, e^{-t})$ at $t = 0$.

c) (5 marks) Let P be the point whose cylindrical coordinates are $(r, \theta, z) = (2, \pi/4, -\sqrt{2})$ and let Q be the point whose spherical coordinates are $(\rho, \theta, \phi) = (2, \pi, \pi/4)$. Compute the distance between P and Q.

a) $r'(t) = (6t, 3-3t^2)$ ← 3 marks

3 mark

$\|r'(t)\| = \sqrt{(6t)^2 + (3-3t^2)^2} = \sqrt{9t^4 + 18t^2 + 9} = 3(t^2+1)$

$s = \int_1^2 (3t^2+3)\,dt = t^3 + 3t \Big]_1^2 = 14 - 4 = \boxed{10}$ ← 4 marks

b) $r'(t) = (\cos t, 1+\sin t, -e^{-t})$ $r'(0) = (1, 1, -1)$ ← 2 marks

$r''(t) = (-\sin t, \cos t, e^{-t})$ $r''(0) = (0, 1, 1)$

2 marks

$r'(0) \times r''(0) = (2, -1, 1)$ ← 2 marks

$\|r'(0)\| = \sqrt{3}$, $\|r'(0) \times r''(0)\| = \sqrt{6}$ ← 2 marks

$K_0 = \frac{\sqrt{6}}{3\sqrt{3}} = \boxed{\frac{\sqrt{2}}{3}}$ ← 2 marks

c) $P = (\sqrt{2}, \sqrt{2}, -\sqrt{2})$ $Q = (-\sqrt{2}, 0, \sqrt{2})$ ← 2 marks

The distance between P and Q is:

2 marks

$\sqrt{(2\sqrt{2})^2 + (\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{18} = \boxed{3\sqrt{2}}$ ← 1 mark