Suslin Lattices

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Outline

Basic Definitions

Definition

*A well founded poset $\langle L, \leq \rangle$ is a **lower semi–lattice** if any $x, y \in L$ have a greatest lower bound $x \wedge y$.*

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*A well founded lower semi–lattice of height ω_1 is said to be **Suslin** if it does not contain an uncountable chain or an uncountable set of pairwise incomparable elements (antichain).*

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- These are lattice analogues of Suslin trees.
- Every (normal) tree is a lower semi–lattice. Thus every Suslin tree is a Suslin lower semi–lattice.

The questions

Suslin lower semi–lattices were considered by Odell in the context of spreading models of Banach spaces. Dilworth, Odell, and Sari [1] asked the following questions:

1. Does ZFC imply that there is a Suslin lower semi–lattice?
2. Does ZFC + CH imply that there is a Suslin lower semi–lattice?
3. Is it consistent to have a Suslin lower semi–lattice but not have any Suslin trees?

Note that a “yes” answer to any question implies a “yes” answer to all suceeding questions.

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- If the requirement that greatest lower bounds exist is dropped, then there is an example in ZFC (the Sierpiński poset).

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- If $\langle L, \leq \rangle$ is not required to be well–founded, but infs. are required to exist, then there are several examples from CH.
- If we concentrate on the case where $\langle L, \leq \rangle$ is Borel (actually all that is needed is for the incomparability relation to be analytic), then by a result of Harrington and Shelah, every Borel poset $\langle L, \leq \rangle$ either has a perfect antichain or is the union of countably many chains.

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Observation

Every $A \subset L$ which has an upper bound has a least upper bound.

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Theorem (Todorčević)

Assume $\mathrm{MA}_{\aleph_1}$. Every uncountable poset $\mathbb{P}$ with no uncountable antichains (comparability) has an uncountable subset each countable subset of which has an upper bound in $\mathbb{P}$.

Corollary

$\mathrm{MA}_{\aleph_1}$ *implies that there are no Suslin lower semi–lattices.*

So the answer to Question 1 is "no".

Partial Answers to Question 2

- The P–ideal dichotomy (PID) is a strong combinatorial principle introduced by Todorčević (see [2]).
- It is a consequence of the Proper Forcing Axiom (PFA), but it is consistent with CH (PFA $\implies \mathfrak{c} = \aleph_2$).
- PID implies that there are no Suslin trees. So it is natural to investigate whether PID implies that there are no Suslin lower semi–lattices as well.

Partial Answers to Question 2

Definition

A (non–principal) ideal $\mathcal{I}$ of countable subsets of ω_1 is called a ***P–ideal*** *if for every countable collection $\{A_n : n \in \omega\} \subset \mathcal{I}$, there exists $A \in \mathcal{I}$ such that $\forall n \in \omega \, [A_n \subset^* A]$.*

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A set $X \subset \omega_1$ is ***orthogonal*** *to $\mathcal{I}$ if $\forall a \in \mathcal{I}\, [|X \cap a| < \aleph_0]$.*

Partial Answers to Question 2

P–Ideal Dichotomy

For every P–ideal $\mathcal{I}$ on ω_1 EITHER

1. *There is an uncountable $X \subset \omega_1$ all of whose countable subsets are in $\mathcal{I}$, OR*
2. $\omega_1 = \bigcup_{n \in \omega} X_n$ *where each X_n is orthogonal to $\mathcal{I}$.*

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A natural candidate for a P–ideal is:

$$\mathcal{I} = \{A \subset L : |A| \leq \aleph_0 \wedge \forall x \in L\, [|\mathrm{pred}\,(x) \cap A| < \aleph_0]\}$$

If $\langle L, \leq \rangle$ is a Suslin tree then $\mathcal{I}$ is a P–ideal.

Partial Answers to Question 2

Theorem

Assume PID *and that there is a Suslin lower semi–lattice. Then there is* $L \subset \mathcal{P}(\omega)$ *such that* $\langle L, \subset \rangle$ *is a Suslin lower semi–lattice, and moreover* $\forall x, y \in L\,[x \wedge y = x \cap y]$.

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- This result is of some interest because it is impossible to have $L \subset \mathcal{P}(\omega)$ such that $\langle L, \subset \rangle$ is a Suslin tree (under any axioms!)

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Theorem

There is a c.c.c. forcing which adds an uncountable set $L \subset \mathcal{P}(\omega)$ *such that* $\langle L, \subset \rangle$ *is a Suslin lower semi–lattice satisfying* $\forall x, y \in L \, [x \wedge y = x \cap y]$. *Moreover* L^n *does not have any uncountable antichains for every* $n \in \omega$.

Partial Answers to Question 2

- There are several constructions from CH of uncountable $L \subset \mathcal{P}(\omega)$ which closed under finite intersections such that $\langle L, \subset \rangle$ has no uncountable antichains.
- For example, Van Douwen and Kunen [3] used CH to construct a sequence $X = \langle a_\alpha : \alpha < \omega_1 \rangle \subset [\omega]^\omega$ such that
  1. $\forall X \in [\omega_1]^{\omega_1} \exists \alpha < \beta \in X \left[a_\alpha \subset a_\beta \right]$
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  2. $\forall \alpha < \beta \left[a_\alpha \subset^* a_\beta \right]$.
- If X is closed under finite intersections, we get L such that $\langle L, \subset \rangle$ has no uncountable antichains, but it is ill founded.

Answer to Question 3

The answer to question 3 is “yes”:

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Idea of proof:

- First force $L \subset \mathcal{P}(\omega)$ such that $\langle L, \subset \rangle$ is a Suslin lower semi–lattice, and moreover L^n has no uncountable antichains for all $n \in \omega$.

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- Now, do a finite support iteration of c.c.c. forcings, and at each stage try to force with $\mathbb{P}(T)$, the usual c.c.c. poset for specializing a given Aronszajn tree T.

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Lemma

If $\mathbb{P}(T)$ adds an uncountable antichain to L^n for some n, there is an uncountable subset $X \subset L^n$ such that forcing with X adds a cofinal branch to T without adding an uncountable antichin to L^m for any m.

Open question: Does $\mathrm{ZFC} + \mathrm{CH}$ imply that there is a Suslin lower semi–lattice?

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