RESEARCH STATEMENT

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1. Past Research

Most of my recent research has focused on the following areas: Tukey theory of ultrafilters on ω, maximal almost disjoint families of sets and functions, preservation theorems for iterated forcing, and the consistency of some consequences of PFA both with CH and with large values of the continuum.

1.1. **Cofinal Types of Ultrafilters.**

We say that a poset $\langle \mathbb{D}, \leq \rangle$ is *directed* if any two members of $\mathbb{D}$ have an upper bound in $\mathbb{D}$. A set $X \subset \mathbb{D}$ is *unbounded* if it doesn't have an upper bound in $\mathbb{D}$. A set $X \subset \mathbb{D}$ is said to be *cofinal* if $\forall y \in \mathbb{D} \exists x \in X \, [y \leq x]$. Given directed sets $\mathbb{D}$ and $\mathbb{E}$, a map $f : \mathbb{D} \to \mathbb{E}$ is called a *Tukey map* if the image (under f) of every unbounded subset of $\mathbb{D}$ is unbounded in $\mathbb{E}$. A map $g : \mathbb{E} \to \mathbb{D}$ is called a *convergent map* if the image (under g) of every cofinal subset of $\mathbb{E}$ is cofinal in $\mathbb{D}$. It is easy to see that there is a Tukey map $f : \mathbb{D} \to \mathbb{E}$ iff there exists a convergent $g : \mathbb{E} \to \mathbb{D}$. When this situation obtains, we say that $\mathbb{D}$ is *Tukey reducible* to $\mathbb{E}$, and we write $\mathbb{D} \leq_T \mathbb{E}$. The relation $\leq_T$ is a quasi order, and induces an equivalence relation in the usual way: $\mathbb{D} \equiv_T \mathbb{E}$ iff both $\mathbb{D} \leq_T \mathbb{E}$ and $\mathbb{E} \leq_T \mathbb{D}$ hold. If $D \equiv_T E$, we say that $\mathbb{D}$ and $\mathbb{E}$ are *Tukey equivalent* or have the same *cofinal type*, and this is intended to capture the idea that $\mathbb{D}$ and $\mathbb{E}$ have "the same cofinal structure".

The notion of Tukey reducibility has proved to be useful in many contexts (see [10], [11], and [2]), particularly in providing a "rough classification" of directed posets. By "rough classification" we mean any classification that is done modulo a similarity type which is coarser than isomorphism type. Such rough classification theorems are useful when one is dealing with a class of structures containing "too many" isomorphism types, so that there can be no meaningful classification theorems modulo isomorphism type for that class.

There are structure theorems as well as non-structure theorems due to Todorčević [37] concerning the possible cofinal types of uncountable directed sets. In the non-structure direction, Todorčević showed that there are $2^{\mathfrak{c}}$ pairwise Tukey inequivalent directed sets of size $\mathfrak{c}$. On the other hand, his structure theorem states that the Proper Forcing Axiom (PFA) implies that there are only five cofinal types of size at most $\aleph_1$: $1, \omega, \omega_1, \omega \times \omega_1$, and $[\omega_1]^{<\omega}$. Here, the ordering on $\omega \times \omega_1$ is the product ordering, and $[\omega_1]^{<\omega}$ is ordered by inclusion.

An ultrafilter $\mathcal{U}$ on ω may be naturally viewed as the directed poset $\langle \mathcal{U}, \supset \rangle$. When this is done, Tukey reducibility turns out to be a coarser quasiorder on ultrafilters than the well studied Rudin-Keisler (RK) reducibility. Recall that an ultrafilter $\mathcal{V}$ is *RK reducible* to $\mathcal{U}$, written $\mathcal{V} \leq_{RK} \mathcal{U}$, if there is a function $f \in \omega^\omega$ such that a set $a \subset \omega$ is in $\mathcal{V}$ iff $f^{-1}(a) \in \mathcal{U}$. There is another motivation for considering the cofinal types in this class of structures. By Todorčević's non-structure result mentioned above, there is no hope of classifying *all* cofinal types of size $\mathfrak{c}$. There are two natural ways to restrict this class. One approach is to demand that the posets be "nicely definable" ideals, and this line is pursued, for example, in [18]. Another, orthogonal, approach is to impose additional structure on the ideals like maximality.

In [31], in joint work with Todorčević, we consider the question of when Tukey reducibility is equivalent to RK reducibility. This is similar in spirit to asking when a automorphism of $\mathcal{P}(\omega)/\operatorname{FIN}$ is induced by a permutation of ω, which was a famous problem in the history of set theory. It is easy to see that when $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters, $\mathcal{V} \leq_T \mathcal{U}$ iff there is a map $\phi : \mathcal{U} \to \mathcal{V}$ such that $\forall a, b \in \mathcal{U} \, [a \subset b \implies \phi(a) \subset \phi(b)]$ and $\forall e \in \mathcal{V} \exists a \in \mathcal{U} \, [\phi(a) \subset e]$. Such maps are said to be *monotone and cofinal*. In [31] we prove:

Theorem 1. *(1) Let $\mathcal{U}$ be an arbitrary ultrafilter and suppose that $\mathcal{V}$ is a Q-point. If there is a continuous monotone and cofinal map $\phi : \mathcal{U} \to \mathcal{V}$, then $\mathcal{V} \leq_{RK} \mathcal{U}$, and moreover, this is witnessed by a finite-to-one function $f \in \omega^\omega$; (2) If $\mathcal{U}$ is selective and $\mathcal{V} \leq_T \mathcal{U}$, then for some $\alpha < \omega_1$, $\mathcal{V} \equiv_{RK} \mathcal{U}^\alpha$, the αth Fubini power of $\mathcal{U}$; (3) Let $\mathcal{K}$ be the class of ultrafilters obtained by closing the P-points under countable Fubini products (i.e. products of the form $\prod_{\mathcal{V}} \mathcal{U}_n$). Suppose $\mathcal{U} \in \mathcal{K}$ and $\mathcal{V}$ is selective. Then if $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V} \leq_{RK} \mathcal{U}$; (4) Assuming CH there exist P-points $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{U} \equiv_T \mathcal{V}$, but $\mathcal{V} <_{RK} \mathcal{U}$; (5) If $\mathcal{U}$ is basically generated (see below) and if $\mathcal{V} \leq_T \mathcal{U}$, then this is always witnessed by a Baire class one monotone and cofinal map $\phi : \mathcal{U} \to \mathcal{V}$.*

The result in (2) gives a complete characterization of the ultrafilters that are Tukey below a selective, and (3) says that $\mathcal{V} \leq_T \mathcal{U}$ implies $\mathcal{V} \leq_{RK} \mathcal{U}$ in many cases where $\mathcal{U} <_T [\mathfrak{c}]^{<\omega}$ and $\mathcal{V}$ is selective. The result in (1) is of some interest because it works even when $\mathcal{U} \equiv_T [\mathfrak{c}]^{<\omega}$ and also because Dobrinen and Todorčević [7] showed that continuous monotone and cofinal maps can always be found when $\mathcal{U}$ is a P-point and $\mathcal{V}$ is any ultrafilter Tukey below it. This result of Dobrinen and Todorčević [7] fails even when $\mathcal{U}$ is the Fubini square of a P-point, but (5) says that the next best thing holds for a wide class of ultrafilters. An ultrafilter $\mathcal{U}$ is said to be *basically generated* if $\mathcal{U}$ has a filter base $\mathscr{B} \subset \mathcal{U}$ with the property that for every $\langle b_n : n \in \omega\rangle \subset \mathscr{B}$ and $b \in \mathscr{B}$, if $\langle b_n : n \in \omega\rangle$ converges to b (with respect to the usual topology on $\mathcal{P}(\omega)$), then there exists $X \in [\omega]^\omega$ such that $\bigcap_{n\in X} b_n \in \mathcal{U}$. Every ultrafilter in the class $\mathcal{K}$ (defined in (3) of Theorem 1) is basically generated.

In joint work with Shelah in [29], we assume MA($\sigma-$ centered) and show how to embed $\langle \mathcal{P}(\omega), \subset^*\rangle$ into the class of P-points equipped with the $\leq_T$ ordering.

1.2. **Almost Disjoint Families.** We say that two infinite subsets a and b of ω are *almost disjoint or a.d.* if $a \cap b$ is finite. We say that a family $\mathscr{A}$ of infinite subsets of ω is *almost disjoint or a.d. in* $[\omega]^\omega$ if its members are pairwise almost disjoint. A *Maximal Almost Disjoint family, or MAD family in* $[\omega]^\omega$ is an infinite a.d. family in $[\omega]^\omega$ that is not properly contained in a larger a.d. family. Two functions f and g in ω^ω are said to be *almost disjoint or a.d.* if they agree in only finitely many places. We say that a family $\mathscr{A} \subset \omega^\omega$ is *a.d. in* ω^ω if its members are pairwise a.d., and we say that an a.d. family $\mathscr{A} \subset \omega^\omega$ is *MAD in* ω^ω if $\forall f \in \omega^\omega \exists h \in \mathscr{A}\, [|f \cap h| = \omega]$. We say that $p \subset \omega \times \omega$ is an *infinite partial function* if it is a function from some $a \in [\omega]^\omega$ to ω. An a.d. family $\mathscr{A} \subset \omega^\omega$ is said to be *Van Douwen* if for any infinite partial function p there is $h \in \mathscr{A}$ such that $|h \cap p| = \omega$. Van Douwen asked whether the existence of a Van Douwen family could be proved in ZFC, and his question was posed as Problem 4.2 in Miller's problem list [21]. In [28], we answer Van Douwen's question, which was open for more than 20 years, by proving

Theorem 2. *There is a Van Douwen MAD $\mathscr{A} \subset \omega^\omega$ of size $\mathfrak{c}$.*

In [24], we answer a question of Shelah and Steprāns [36] that is closely related to the metrization problem for countable Fréchet groups. Let FIN denote the non-empty finite subsets of ω. Given an ideal $\mathcal{I}$ on ω, we say that $P \subset$ FIN is *$\mathcal{I}$-positive* if $\forall a \in \mathcal{I} \exists s \in P\, [a \cap s = 0]$. Given an a.d. family $\mathscr{A} \subset [\omega]^\omega$, let $\mathcal{I}(\mathscr{A})$ denote the ideal on ω generated by $\mathscr{A}$. We say that an a.d. family $\mathscr{A} \subset [\omega]^\omega$ is *strongly separable* if for each $\mathcal{I}(\mathscr{A})$-positive $P \subset$ FIN, there is $a \in \mathscr{A}$ and $Q \in [P]^\omega$ such that $\bigcup Q \subset a$. Thus this notion is gotten from the well known notion of a completely separable a.d. family by replacing integers with finite sets in the definition. Shelah [33] has recently proved that completely separable a.d. families exist if $\mathfrak{c} < \aleph_\omega$. But in [24], we prove that strong separability behaves differently.

Theorem 3. *It is consistent that there are no strongly separable a.d. families and $\mathfrak{c} = \aleph_2$.*

An a.d. $\mathscr{A} \subset [\omega]^\omega$ such that for each countable collection of sets $\{a_n : n \in \omega\} \subset \mathcal{P}(\omega) \setminus \mathcal{I}(\mathscr{A})$, $\exists c \in \mathscr{A} \forall n \in \omega\, [|c \cap a_n| = \omega]$ is called *tight* or *$\aleph_0$-MAD*. It is a long standing open problem whether such a family can be constructed in ZFC. Hrušák and García Ferreira [15] introduced a natural weakening of this notion, and applied it to the Katétov ordering on MAD families. An a.d. family $\mathscr{A} \subset [\omega]^\omega$ is *weakly tight* if for each countable collection of sets $\{a_n : n \in \omega\} \subset \mathcal{P}(\omega) \setminus \mathcal{I}(\mathscr{A})$, there exists $c \in \mathscr{A}$ such that for infinitely many $n \in \omega$, $|c \cap a_n| = \omega$. In joint work with Steprāns in [30], we modify the recent techniques of Shelah [33] alluded to above to prove

Theorem 4. *If $\mathfrak{s} \leq \mathfrak{b} < \aleph_\omega$, then there is a weakly tight MAD family $\mathscr{A} \subset [\omega]^\omega$.*

A notable novel feature of the construction in Shelah [33] and our construction in [30] is the use of techniques from PCF theory to build a set of reals.

1.3. **Preservation theorems.** In [26], we answer a well known question of Kellner and Shelah by proving the following: Let γ be a limit ordinal and let $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma\rangle$ be a countable support (CS) iteration. Suppose that for each $\alpha < \gamma$, $\Vdash_\alpha$ "$\mathring{\mathbb{Q}}_\alpha$ is proper" and that $\mathbb{P}_\alpha$ does not turn $\mathbf{V} \cap \omega^\omega$ into a meager set. Then $\mathbb{P}_\gamma$ does not do so either.

In [25] we have some partial results towards a proof of the analogous result for the property of not turning $\mathbf{V} \cap \omega^\omega$ into a null set.

1.4. **Suslin Lattices.** A well founded poset $\langle L, \leq\rangle$ is a *lower semi-lattice* if every $x, y \in L$ have a greatest lower bound $x \wedge y$. We say that a well founded lower semi-lattice is *Suslin* if it is uncountable, but does not contain an uncountable chain or an uncountable set of pairwise incomparable elements. Note that any normal Suslin tree is a Suslin lower semi-lattice. It is well known that CH does not imply the existence of a Suslin tree. However, the corresponding question for Suslin lower semi-lattices is still open. In joint work with Yorioka in [8], we provide some partial results on this question. In particular, we show that if CH implies the existence of a Suslin lower semi-lattice, then it also implies the existence of one that is a substructure $\langle \mathcal{P}(\omega), \subset, \cap\rangle$. We also examine some consequences of Todorčević's P-ideal dichotomy for Suslin lower semi-lattices that are substructures of $\langle \mathcal{P}(\omega), \subset, \cap\rangle$.

1.5. **Gregory Trees.** View the tree $2^{<\omega_1}$ as a forcing poset, defining $p \leq q$ iff $p \supset q$.A *Cantor tree* in $2^{<\omega_1}$ is a subset $\{f_\sigma : \sigma \in 2^{<\omega}\} \subset 2^{<\omega_1}$ such that for all $\sigma \in 2^{<\omega}$, $f_{\sigma^\frown 0}$ and $f_{\sigma^\frown 1}$ are incompatible nodes in $2^{<\omega_1}$ that extend f_σ. A subtree $\mathbb{T}$ of $2^{<\omega_1}$ has the *Cantor Tree Property (CTP)* if 1) For every $f \in \mathbb{T}$, $f^\frown 0, f^\frown 1 \in \mathbb{T}$, and 2) Given any Cantor tree $\{f_\sigma : \sigma \in 2^{<\omega}\} \subset \mathbb{T}$, there are $x \in 2^\omega$ and $g \in \mathbb{T}$ such that $\forall n \in \omega\,[g \leq f_{x\restriction n}]$. A subtree $\mathbb{T}$ of $2^{<\omega_1}$ is a *Gregory tree* if it has the CTP, but does not have a cofinal branch.

Gregory [12] showed that $2^{\aleph_0} < 2^{\aleph_1}$ implies that there is a Gregory tree. On the other hand, PFA implies that there are no Gregory trees. Whenever a result is proved from PFA, two natural questions arise. First, does it follow just from MA + ¬CH? Second, is it consistent with $2^{\aleph_0} > \aleph_2$? In joint work with Kunen, we show in [17] that the answer is "no" to the first question and "yes" to the second.

2. Future Research Plans

Our research will explore the connections between iterated forcing and set theory of the continuum. We will focus on the following topics: consistency with the continuum hypothesis (CH), existence of almost disjoint families with strong combinatorial properties, Michael's problem, Tukey theory of ultrafilters on ω, preservation theorems for iterated forcing, and Naimark's problem. We expect our research to yield a better understanding of the implications of CH for the structure of the filter of closed unbounded (*club*) subsets of ω_1. We expect to discover more powerful ways to use cardinal invariants of the continuum in combinatorial constructions that require a diagonalization of length continuum (for a survey of existing uses see [27]). We also hope to understand better under what circumstances iterated forcing does not change the continuum "too much", both in the sense of not adding any new reals, and also in the sense of not adding reals of a specific kind. Finally, we expect to discover fresh applications of set theory to operator algebras.

2.1. CH **and iterated forcing.** A prominent theme in set theory is the investigation of forcing axioms and the combinatorial principles that are their consequences. Forcing axioms, such as the Proper Forcing Axiom (PFA), and combinatorial dichotomies, such as the P-ideal dichotomy (PID), are consistent with ZFC and are known to settle a wide variety of questions that are not resolvable in ZFC, ranging from the value of the continuum to questions about automorphisms of the Calkin algebra. By contrast, much less progress has been made on developing a comparable array of axioms and principles that are consistent with ZFC + CH and answer questions left open by this theory. Indeed, by recent results of Aspero, Larson, and Moore, there can be no "optimal" forcing axiom that is consistent with ZFC+CH.

One natural way to proceed is to try to *discover how much of* PFA *and Martin's Maximum (*MM*) are consistent with* CH. For this, it is important to have preservation theorems that tell us when an iteration of proper posets which do not add any new reals itself does not add new reals. Let us call a proper poset that does not add reals *totally proper*. Devlin and Shelah [6] discovered a fundamental limitation to such preservation theorems, known as the weak diamond, which is a consequence of CH, even though each witness to it can be destroyed by a totally proper forcing. The notion of $\mathbb{D}$-completeness was introduced by Shelah [35] as "medicine" against this limitation. We give a somewhat informal description of it here. A "nicely definable" function $\mathbb{D}$ that assigns to each countable $M \prec H(\theta)$, $\mathbb{P} \in M$, and $p \in M \cap \mathbb{P}$ a non-empty, pairwise intersecting collection, $\mathbb{D}(M, \mathbb{P}, p)$, of sets of $(M, \mathbb{P})$ generic filters that contain p is called a *simple, 2-complete completeness system*. A poset $\mathbb{P}$ is said to be $\mathbb{D}$*-complete* if for every countable $M \prec H(\theta)$ with $\mathbb{P} \in M$ and every $p \in M \cap \mathbb{P}$, there is an $X \in \mathbb{D}(M, \mathbb{P}, p)$ such that each member of X has a lower bound in $\mathbb{P}$. We call $\mathbb{P}$ *iterable* if there exists some simple, 2-complete completeness system $\mathbb{D}$ such that $\mathbb{P}$ is $\mathbb{D}$-complete.

Shelah [35] showed that the a countable support (CS) iteration of posets that are iterable and are α proper for every $\alpha < \omega_1$ does not add reals, and thus proved the consistency of the natural forcing axiom for such posets with CH. Although this axiom settles many questions left open by ZFC + CH (see, for example, Todorčević [38]), there is an important class of statements which it does not decide, and this failure is closely tied to the requirement of α properness in Shelah's iteration theorem. Some of these statements can be tackled using another preservation theorem due to Shelah [35]: A CS iteration of posets that are iterable does not add reals *provided* each iterand remains proper after forcing with an arbitrary totally proper poset. But there are several statements not addressed by either theorem. It is known that the requirement of α properness cannot be dropped from Shelah's first iteration theorem if $\mathbf{V} = \mathbf{L}$ is assumed. So our first question aims to probe the consistency of the strongest possible iteration principle for not adding reals: Assuming the existence of large cardinals, is it true that any CS iteration of forcings that are iterable is totally proper? The large cardinals are relevant here because the above counterexample uses properties of $\mathbf{L}$ that badly fail in $\mathbf{V}$ if there is, say, a measurable cardinal.

As a test question, Moore formulated the following statement, which may be seen as the ultimate failure of club guessing. For a set $X \subset \omega_1$, $\mathrm{Lim}\,(X)$ denotes the set of limit points of X. Let $\alpha \in \mathrm{Lim}(\omega_1)$, and let $a, b \subset \alpha$ be closed unbounded subsets of α. We say that *a measures b* if there is a $\beta < \alpha$ such that *either* $(a \setminus \beta) \subset b$ *or* $(a \setminus \beta) \cap b = \emptyset$. The following statement is called *measuring*: For every sequence $\langle c_\alpha : \alpha \in \mathrm{Lim}\,(\omega_1)\rangle$, where $c_\alpha \subset \alpha$ is a closed unbounded subset of α, there is a club $E \subset \omega_1$ such that for each $\alpha \in \mathrm{Lim}(E)$, $E \cap \alpha$ measures c_α. PFA implies measuring. We will investigate the question "Is measuring consistent with CH?" This is a good test question for finding better iteration theorems. The poset for forcing an instance of measuring, though iterable, is not ω proper, and its properness may be destroyed even by $2^{<\omega_1}$, the most innocuous of totally proper orders, so that some essentially new type of preservation theorem will be needed. Moreover, most weakenings of measuring are known to be consistent with CH. There is also an interesting connection between this question and the deep problem of whether saturation is consistent with CH. *Saturation* is the proposition that whenever $\{S_\alpha : \alpha < \omega_2\}$ is a collection of stationary subsets of ω_1, there exist $\beta < \alpha < \omega_2$ such that $S_\beta \cap S_\alpha$ is stationary; saturation is a celebrated consequence of MM (see [9]). Woodin [39] showed that if there is a measurable cardinal, then saturation contradicts CH, and it is suspected that large cardinals are necessary for this conclusion. For a poset $\mathbb{P}$, $\mathrm{FA}^+(\mathbb{P})$ is the following statement: For any collection $\langle D_\alpha : \alpha < \omega_1\rangle$ of dense open subsets of $\mathbb{P}$, and for each $\mathbb{P}$ name $\mathring{S}$ for a stationary subset of ω_1, there is a filter $G \subset \mathbb{P}$ such that for each $\alpha < \omega_1$, $G \cap D_\alpha \neq \emptyset$, and $\{\alpha : \exists p \in G[p \Vdash \alpha \in \mathring{S}]\}$ is stationary in ω_1. It is not difficult to show that $\mathrm{FA}^+(2^{<\omega_1})$ together with saturation implies measuring. This is of significance because if $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \delta\rangle$ is *any* revised countable support iteration of non-trivial, stationary preserving posets such that $\mathbb{P}_\delta$ does not add new ω sequences of ordinals, and if $\Vdash_\delta \mathrm{cf}(\delta) \geq \omega_2$, then $\mathbb{P}_\delta$ *automatically* forces $\mathrm{FA}^+(2^{<\omega_1})$. Therefore, any conventional attempt to produce a model of saturation + CH must first address the simpler problem of producing a model of measuring + CH via such an iteration. On the other hand, a proof that measuring refutes CH will show that it is impossible to produce a model of saturation + CH by these means. Such an implication of CH for the structure of the club filter on ω_1 would be fundamental.

A possible approach to showing that measuring is consistent with CH could be to exploit the fact that the poset for forcing an instance of measuring is "nicely definable". This, especially in the presence of large cardinals, may help in showing that the iteration of these posets does not add reals. My research in [25] and [26] on preservation theorems for CS iterations and my current research in [32] on Suslin lattices are also relevant to the investigation of these questions.

We end with two questions, the first one well known, the other technical. Recall that a topological space X is *countably tight* if whenever $x \in X$ is in the closure of a set $Y \subset X$, we can find a countable $Z \subset Y$ which has x in its closure. X is *sequential* if a set $Y \subset X$ is closed iff Y contains all limits of convergent sequences from Y. A famous question, known as the Moore-Mrówka problem, asks if every compact Hausdorff space which is countably tight is sequential. The answer is "no" under $\diamond$ and "yes" under PFA; it is unknown what happens under CH, with the poset for destroying a counterexample falling into the same category as the poset for measuring. Now, we state the technical question: Suppose $\mathbb{P}$ is a CS iteration of posets that are iterable, and suppose $\mathbb{P}$ adds reals, does $2^{<\omega_1} \times \mathbb{P}$ collapse ω_1?

2.2. **Almost disjoint families.** One natural property of interest of maximal objects (of any sort) is preservation of their maximality in forcing extensions. Let $\mathbb{P}$ be a notion of forcing and let $\mathscr{A}$ be a MAD family either in $[\omega]^\omega$ or ω^ω (We refer the reader to 1.2 for the definitions). We say that $\mathscr{A}$ *is* $\mathbb{P}$*-indestructible* if $\Vdash_{\mathbb{P}} \mathscr{A}$ is MAD. There is no forcing notion $\mathbb{P}$ adding a new real for which we know how to construct a $\mathbb{P}$-indestructible MAD family in ZFC. A Sacks indestructible MAD family is provably the weakest such object in the sense that if $\mathscr{A}$ is $\mathbb{P}$-indestructible for some $\mathbb{P}$ that adds a real, then $\mathscr{A}$ is also Sacks indestructible. It is known that if $\mathfrak{a} < \mathfrak{c}$, then any MAD family of size $\mathfrak{a}$ is automatically Sacks indestructible, and it is known that Sacks indestructible MAD families of size $\mathfrak{c}$ exist if either $\mathfrak{b} = \mathfrak{c}$ or $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ holds (see [4] and [14]). Turning to stronger notions of indestructibility, it turns out that there is a Cohen indestructible MAD family in $[\omega]^\omega$ iff there is an $\aleph_0$-MAD family in $[\omega]^\omega$ (see 1.2). Malykhin [19] asked in 1989 whether such families exist; it is known that the answer is yes if $\mathfrak{b} = \mathfrak{c}$. Weakly tight families, though not indestructible, are a natural weakening of $\aleph_0$ MAD families, and are known to exist if $\mathfrak{s} \leq \mathfrak{b} < \aleph_\omega$ (see 1.2). In particular, weakly tight families exist if $\mathfrak{b} = \mathfrak{d}$.

We will investigate the existence as well as possible cardinalities of $\mathbb{P}$-indestructible MAD families. We will do this both for MAD families in $[\omega]^\omega$ and in ω^ω, and for a range of posets $\mathbb{P}$, including Sacks, Cohen, and Miller forcings. We will also investigate the existence of weakly tight families. The techniques recently introduced by Shelah [33] in his construction of a completely separable MAD family under $\mathfrak{c} < \aleph_\omega$ and their modification in [30] are especially relevant here. We conjecture that a further modification of [30] will yield a Sacks indestructible MAD family from $\mathfrak{s} \leq \mathfrak{b} < \aleph_\omega$. On the other hand, the construction in [30] suggests that there could be a model of $\mathfrak{b} < \mathfrak{s}$ with no $\aleph_0$ MAD families. Therefore, we will investigate whether the various partial orders that have been independently devised by Brendle, Canjar, and Shelah (see [3], [5], and [34]) for forcing $\mathfrak{b} < \mathfrak{s}$ can be modified to achieve this. My work on the indestructibility properties of $\aleph_0$ MAD families in ω^ω in [26], and on Van Douwen families in [28] is also relevant.

With a few notable exceptions, extant constructions of a.d. families with strong combinatorial properties do not make sharp use of cardinal invariants, often using assumptions of the form $\mathfrak{x} = \mathfrak{c}$. We expect techniques for the deeper use of cardinal invariants to emerge from our investigation, so that assumptions of the form $\mathfrak{x} \leq \mathfrak{y}$ will suffice. This is likely to be of use in other contexts as well, such as the metrization problem for countable Fréchet groups.

Recall that a topological space X is *Fréchet* if whenever a point $p \in X$ is in the closure of a set $A \subset X$, there is a sequence of points in A converging to p. A well-known question of Malykhin asks whether every countable Fréchet group is metrizable. Let us say that an ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ is *Fréchet* if for every $\mathcal{I}$-positive $P \subset \mathrm{FIN}$ (see 1.2), there is a $Q \in [P]^{\omega}$ consisting of pairwise disjoint sets so that $\forall a \in \mathcal{I}\, [|a \cap (\bigcup Q)| < \omega]$. Clearly, this notion is closely related to the notion of a strongly separable a.d. family (see 1.2). If $\mathcal{I}$ is a Fréchet ideal which is not countably generated, then we can define a non-metrizable Fréchet topology on $\langle [\omega]^{<\omega}, \triangle \rangle$, where $\triangle$ denotes symmetric difference, by stipulating that $\{A \subset [\omega]^{<\omega} : \exists a \in \mathcal{I}\, [[\omega \setminus a]^{<\omega} \subset A]\}$ is a neighborhood base at 0. We hope to investigate the following questions of Gruenhage and Szeptycki [13]: Is there an uncountable a.d. family $\mathscr{A} \subset [\omega]^{\omega}$ such that $\mathcal{I}(\mathscr{A})$ is Fréchet? Is there a Fréchet ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ that is not countably generated?

2.3. **Michael's Problem.** A problem of Michael [20] dating back to 1971 asks whether there is a regular Lindelöf space whose product with the space of irrationals (say $[\omega]^{\omega}$ with the usual topology) is not Lindelöf. Such a space is called a *Michael space*. As regular Lindelöf spaces are normal, this may be seen as a variation on the famous Dowker space problem, which asks for a normal space whose product with the unit interval is not normal. It is easy to see that if X is Lindelöf, then any open cover of $X \times [\omega]^{\omega}$ has a sub-cover of size at most $\mathfrak{d}$. Let FIN denote the finite subsets of ω. We say that a sequence $\langle X_\alpha : \alpha \leq \theta \rangle$ is a *θ-Michael sequence* if the following hold: 1) for each $\beta < \alpha \leq \theta$, $\mathrm{FIN} = X_\theta \subsetneq X_\alpha \subsetneq X_\beta \subset \mathcal{P}(\omega)$; 2) for each compact $K \subset [\omega]^{\omega}$, $\min\{\xi \leq \theta : K \cap X_\xi = \emptyset\}$ does not have uncountable cofinality. Moore [22] showed that if there is a θ-Michael sequence for some cardinal θ of uncountable cofinality, then there is a Michael space, and that if X is a Michael space and $\mathcal{U}$ is an open cover of $X \times [\omega]^{\omega}$ with no sub-cover of smaller cardinality, then there is a $|\mathcal{U}|$-Michael sequence. In particular, when $\mathfrak{c} < \aleph_\omega$, there is a Michael space iff there is a θ-Michael sequence for some uncountable $\theta \leq \mathfrak{d}$. Moore [22] showed that if $\mathfrak{d} = \mathrm{cov}(\mathcal{M})$, then there is a $\mathfrak{d}$-Michael sequence. We will investigate the existence of θ-Michael sequences. The most natural way to proceed is to see if they exist in the Laver and Mathias models as those are our best understood models of $\mathrm{cov}(\mathcal{M}) < \mathfrak{d}$, and the situation there is unknown.

2.4. **Cofinal types of ultrafilters.** The most outstanding problem in the Tukey theory of ultrafilters on ω is the following: Is it consistent that for every ultrafilter $\mathcal{U}$ on ω, $\langle \mathcal{U}, \supset \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subset \rangle$? This question, due to Isbell [16], dates to 1965. It is equivalent to asking whether it is consistent that for each ultrafilter $\mathcal{U} \subset \mathcal{P}(\omega)$, there is a set $\{a_\alpha : \alpha < \mathfrak{c}\} \subset \mathcal{U}$ such that for each $x \in [\mathfrak{c}]^{\aleph_0}$, $\bigcap_{\alpha \in x} a_\alpha \notin \mathcal{U}$. A positive result would be striking because it would say that consistently, all ultrafilters on ω are the same in the sense of cofinal type. Whereas a negative solution would show how to build an ultrafilter with a certain degree of "P-point-ness" in ZFC. A closely related problem is the following. Are there ultrafilters $\mathcal{U} <_T [\mathfrak{c}]^{<\omega}$ and $\mathcal{V} <_T [\mathfrak{c}]^{<\omega}$ so that $\mathcal{U} \times \mathcal{V} \equiv_T [\mathfrak{c}]^{<\omega}$?

2.5. **Preservation Theorems for Iterated Forcing.** Preservation theorems are an important component of the theory of iterated forcing. The simplest and most well–known preservation theorem says that the finite support iteration of c.c.c. posets is c.c.c. For CS iterations, a fundamental result of Shelah [35] is that properness is preserved under such iterations. For a property $\mathbf{P}$ of forcing notions, two kinds of preservation under CS iterations may be considered. One kind says if $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is a CS iteration such that for each $\alpha < \gamma$, $\Vdash_\alpha$ "$\mathring{\mathbb{Q}}_\alpha$ is proper and has property $\mathbf{P}$", then $\mathbb{P}_\gamma$ also has property $\mathbf{P}$. An example of this kind is the preservation of ω^ω–bounding (see [35]). The other kind of preservation theorems say that given a limit ordinal γ, if $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is a CS iteration such that for each $\alpha < \gamma$, $\Vdash_\alpha$ "$\mathring{\mathbb{Q}}_\alpha$ is proper" and $\mathbb{P}_\alpha$ has property $\mathbf{P}$, then $\mathbb{P}_\gamma$ also has property $\mathbf{P}$. An example is the preservation of the property of not making $\mathbf{V} \cap \omega^\omega$ meager proved in [26]. One often considers a preservation theorem of the second kind only for properties that fail to be preserved by two step iterations, so that there can be no result of the first kind for them. We plan to investigate the two most outstanding questions on preservation theorems of the second kind: Let γ be a limit ordinal and let $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ be a CS iteration. Suppose that for each $\alpha < \gamma$, $\Vdash_\alpha$ "$\mathring{\mathbb{Q}}_\alpha$ is proper" and $\mathbb{P}_\alpha$ does not add a Cohen real. Is it true that $\mathbb{P}_\gamma$ also does not add a Cohen real? Does preservation of the second kind hold for the property of not turning $\mathbf{V} \cap \omega^\omega$ into a null set? Some partial results on the second question may be found in [25].

2.6. **Naimark's problem.** Let H be an infinite dimensional, complex, not necessarily separable, Hilbert space. $\mathcal{B}(H)$ denotes the algebra of bounded operators on H and $\mathcal{K}(H)$ is its ideal of compact operators. The *Calkin algebra*, denoted $C(H)$, is $\mathcal{B}(H)\ /\ \mathcal{K}(H)$. Many applications of set theory to the Calkin algebra have been seen recently. For example, Phillips and Weaver [23] showed that the Calkin algebra of a separable H has outer automorphisms under CH, while Farah [8] showed that all of its automorphisms are inner under PFA. A long standing open question of Naimark asks whether every C^* algebra that has a unique irreducible representation up to unitary equivalence is isomorphic to $\mathcal{K}(H)$ for some H. Akemann and Weaver [1] proved that the answer is "no" under $\diamond$, and it is conceivable that the answer is independent of ZFC. We hope to investigate whether the answer to Naimark's question is "yes" under PFA, and in various forcing extensions.

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