

ALMOST DISJOINT FAMILIES AND DIAGONALIZATIONS OF LENGTH CONTINUUM

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Abstract. We present a survey of some results and problems concerning the existence of almost disjoint families all of which require a construction involving a diagonalization of length continuum to be carried out. We emphasize the role of cardinal invariants of the continuum and their combinatorial characterizations in such constructions.

§1. Introduction. The phrase “diagonalization of length continuum” is used in set theory to refer to recursive constructions during which continuum many “requirements” must be met. Such arguments go back to the beginnings of set theory. Although the construction of any type of object may require a diagonalization of length continuum, this obligation is most often encountered during the construction of sets of reals that satisfy certain combinatorial properties. Consider, for example, the familiar construction of a Bernstein set, a set of reals such that neither it nor its complement contains a perfect set. One enumerates the continuum many perfect sets in type \mathfrak{c} , say as $\langle P_\alpha : \alpha < \mathfrak{c} \rangle$. Then one constructs the Bernstein set X in \mathfrak{c} steps. At stage α , one diagonalizes against P_α , ensuring that neither X nor its complement contains P_α . It is possible to do this since, at stage α , fewer than \mathfrak{c} reals have been put into X or its complement, while P_α has size \mathfrak{c} .

The Bernstein set is an example of a diagonalization of length \mathfrak{c} that can be carried out in ZFC. However, many such diagonalizations require additional hypotheses and it is, in general, not possible to tell when such a diagonalization can be done in ZFC alone. One of the key issues involves the smaller diagonalizations that must be performed at each stage of the main diagonalization. In the case of the Bernstein set, at stage α ,

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we need to diagonalize against the reals that have already been placed into X or into its complement to produce two fresh reals in P_α that are different from all of them. This is a diagonalization of length less than \mathfrak{c} , and a typical feature of diagonalizations of length \mathfrak{c} is that such smaller diagonalizations must be performed at each stage of the main diagonalization. We will refer to these smaller diagonalizations as *sub-diagonalizations* in this article. For the Bernstein set construction, the sub-diagonalizations proceed at each stage because perfect sets have size \mathfrak{c} . In general, however, one needs to make further assumptions about the continuum in order to ensure that the sub-diagonalizations can be done. The most common such assumption is CH, and this suffices for essentially all diagonalizations of length \mathfrak{c} . There is a deep fact underlying this. Let us say that a statement is Σ_1^2 if it has the form $\exists X \subset \mathcal{P}(\omega)\psi(X)$, where $\psi(X)$ is an arbitrary second order formula over the structure $(\omega, <, +, \times)$. In all diagonalizations of length \mathfrak{c} that interest us, each of the \mathfrak{c} many “requirements” can be coded as a real number, and the statement that there is a set of reals “meeting all the requirements” rephrased as, “there is a set $X \subset \mathcal{P}(\omega)$ such that for every real $r \in \mathcal{P}(\omega) \dots$ ”. Here the quantifier “for every real $r \in \mathcal{P}(\omega)$ ” is ranging over the \mathfrak{c} requirements that X must fulfill. Therefore, a diagonalization of length \mathfrak{c} proves a statement of the form $\exists X \subset \mathcal{P}(\omega)\forall r \in \mathcal{P}(\omega)\phi(X, r)$, which is an example of a Σ_1^2 statement. A deep result of Woodin states that relative to the consistency of large cardinals, any Σ_1^2 statement that can be forced follows from CH. This means that if the end product of a diagonalization of length \mathfrak{c} can be forced to exist, which will almost certainly be the case if its existence is at all consistent, then its existence already follows from CH. Here is the precise statement of Woodin’s theorem; for more details see [12].

THEOREM 1.1 (Woodin). *Assume that there are class many measurable Woodin cardinals. If a Σ_1^2 statement ϕ is true in some forcing extension, then it is true in any forcing extension satisfying CH.*

In fact, it turns out that CH facilitates these diagonalizations in two different ways: one by allowing the sub-diagonalizations required at each stage to be carried out, and second, by providing a small set of reals that “captures” the continuum in some way. CH achieves both goals simultaneously by the making the entire continuum small. In this article, we will classify the diagonalizations of length \mathfrak{c} into two types depending on whether they need both kinds of help from CH or just the first kind. There is a fundamental distinction between the two, and it is desirable to distinguish between them. We will begin by describing a famous example of each kind of diagonalization of length \mathfrak{c} . Then in Section 2 we will point out that cardinal invariants of the continuum can be used to separate the two kinds of diagonalizations, and in the remainder of the article, we will focus exclusively on diagonalizations of the first kind. We

will describe some constructions of almost disjoint families requiring a diagonalization of length \mathfrak{c} that can be carried out in ZFC. We will first describe a well-known example due to Balcar and Vojtáš [4] and then a more recent example due to the author [24]. The unifying theme in these examples is that a diagonalization of length \mathfrak{c} that appears to require additional hypotheses *a priori* may be doable in ZFC through the use of deep combinatorial characterizations of appropriate cardinal invariants. Finally, in Section 6, we will end with some open questions regarding whether certain constructions of almost disjoint families that require a diagonalization of length continuum can be carried out in ZFC.

We begin with an example of a diagonalization of length \mathfrak{c} of the first kind, namely one which requires an additional hypothesis only for a sub-diagonalization at each stage. Recall that a non-empty set $\mathcal{F} \subset \mathcal{P}(\omega)$ is a *filter on ω* if \mathcal{F} is closed under supersets and pairwise intersections, but does not contain any finite sets. Recall also that an *ultrafilter on ω* is a maximal filter on ω – i.e. a filter such that $\forall a \subset \omega [a \in \mathcal{F} \vee (\omega \setminus a) \in \mathcal{F}]$; any filter on ω can be extended to an ultrafilter using Zorn’s lemma. A famous problem in the history of set theory concerns the existence of a certain kind of ultrafilter on ω whose construction requires a diagonalization of length \mathfrak{c} . We say that b is *almost contained* in a , and we write $b \subset^* a$, if $b \setminus a$ is finite. An ultrafilter \mathcal{U} is called a *P-point* if for every countable family $\{a_n : n \in \omega\} \subset \mathcal{U}$, there is $b \in \mathcal{U}$ such that $\forall n \in \omega [b \subset^* a_n]$. P-points were intensively studied in connection with the question of whether $\beta\omega$ is homogeneous, and an historically important question in set theory was whether P-points exist. Let us consider what is involved in constructing a P-point. We must diagonalize all countable subsets of the ultrafilter we are building (in addition to ensuring that we do end up with an ultrafilter). As there are \mathfrak{c} such subsets, we need to do a diagonalization of length continuum. We cannot enumerate all the countable subsets of the ultrafilter at the start of the construction; so we instead list all countable subsets of $[\omega]^\omega$ in type \mathfrak{c} with each countable subset occurring \mathfrak{c} times in the list. This is to ensure that each countable subset will have sufficiently many opportunities to be taken care of. Also, let $\langle c_\alpha : \alpha < \mathfrak{c} \rangle$ enumerate $[\omega]^\omega$. At any stage $\alpha < \mathfrak{c}$, we have a filter \mathcal{F}_α that is generated by fewer than \mathfrak{c} sets, some countable family $\{a_n : n \in \omega\} \subset [\omega]^\omega$, and a set $c_\alpha \in [\omega]^\omega$. To make sure we end up with an ultrafilter, we need to decide whether to put c_α or its complement into $\mathcal{F}_{\alpha+1}$; but this is easy to take care of and does not require any sub-diagonalization. The difficulty is in dealing with $\{a_n : n \in \omega\}$; c_α can be dealt with after that. If $\{a_n : n \in \omega\} \not\subset \mathcal{F}_\alpha$, then we do nothing; but if $\{a_n : n \in \omega\} \subset \mathcal{F}_\alpha$, then we have to perform a sub-diagonalization to produce a $b \in [\omega]^\omega$ which is positive for the filter \mathcal{F}_α such that $\forall n \in \omega [b \subset^* a_n]$. To be positive for \mathcal{F}_α , it is enough for b to have infinite intersection with all the members of some

filter base for \mathcal{F}_α , which we know can be chosen to be of size less than \mathfrak{c} . So our sub-diagonalization is of length less than \mathfrak{c} , and we are required to have constructed \mathcal{F}_α in such a way that this sub-diagonalization is possible. But unlike in the case of a Bernstein set, it is far from clear that there are any “right choices” that we could have made at stages before α , either regarding the \mathfrak{c} s or the earlier sub-diagonalizations, to guarantee that \mathcal{F}_α is such that the sub-diagonalization required at stage α can be carried out. Indeed Shelah constructed a model of set theory in which P-points don’t exist (see Theorem 4.4.7 of [7]). Nevertheless, note that the sub-diagonalization can be done if \mathcal{F}_α is countably generated, and that this will be the case for every α provided $\mathfrak{c} = \omega_1$. Thus CH allows us to build a P-point by making the necessary sub-diagonalizations be only of countable length.

Now, we give a well-known example of a diagonalization of length \mathfrak{c} that requires extra hypotheses both to guarantee the existence of a small set of reals “capturing” the continuum in some way, and to perform a sub-diagonalization at each stage. Whether S and L spaces exist was a famous problem in the history of set-theoretic topology. We will not deal directly with this problem, but with another problem about sets of reals which is very closely related to it. Let us say that a family $X \subset [\omega]^\omega$ is *well founded* if the poset $\langle X, \subset \rangle$ has no infinite descending chains – i.e. if there is no sequence $\langle a_n : n \in \omega \rangle \subset X$ so that $a_{n+1} \subsetneq a_n$. Given $X \subset [\omega]^\omega$, we say a family $A \subset X$ is an antichain if $\forall a, b \in A [a \neq b \implies (a \not\subset b \wedge b \not\subset a)]$. We are interested in the question of whether there is an uncountable well founded family $X \subset [\omega]^\omega$ with no uncountable antichains. First of all notice that if any such a family exists, then there is one of size ω_1 . Notice also that a well founded family $X \subset [\omega]^\omega$ of size ω_1 can be enumerated as $\langle x_\alpha : \alpha < \omega_1 \rangle$ in such a manner that $\forall \alpha, \beta < \omega_1 [x_\alpha \subset x_\beta \implies \alpha \leq \beta]$. Readers familiar with S and L spaces will immediately see that such a family $X = \langle x_\alpha : \alpha < \omega_1 \rangle$ is a right separated subspace of $[\omega]^\omega$ when endowed with the Vietoris topology. This is the topology on $[\omega]^\omega$ generated by sets of the form $[s, a] = \{b \in [\omega]^\omega : s \subset b \subset a\}$, where $s \in [\omega]^{<\omega}$ and $a \in [\omega]^\omega$. Moreover, not containing any uncountable antichains, X has no uncountable discrete subspaces. Therefore, in the Vietoris topology, X is an example of an S space which is moreover first countable, since the Vietoris topology is first countable. This is because for each $a \in [\omega]^\omega$, $\{[s, a] : s \in [a]^{<\omega}\}$ is a local base at a .

Let us now return to the purely combinatorial question of whether there exists a family $X = \langle x_\alpha : \alpha < \omega_1 \rangle \subset [\omega]^\omega$ such that for all $\alpha < \beta < \omega_1$, $x_\beta \not\subset x_\alpha$, which doesn’t contain any uncountable antichains. At first this may not seem like a diagonalization of length \mathfrak{c} because to ensure that X contains no uncountable antichains, we seem to need to diagonalize against its uncountable subsets, which are 2^{\aleph_1} in number. However, Van

Douwen and Kunen [30] first observed that only 2^{\aleph_0} requirements need to be met if we replace the condition of not containing any uncountable antichains with the following stronger requirement:

- for each countable set $A \subset \omega_1$, there is $\alpha < \omega_1$ such that for
- (*) all $\beta \geq \alpha$, if x_β is in the closure (with respect to the usual topology on $[\omega]^\omega$) of $\{x_\gamma : \gamma \in A\}$, then $\exists \gamma \in A [x_\gamma \subset x_\beta]$.

Let us first see that requirement (*) is indeed stronger than not containing any uncountable antichains. Let $Y \in [\omega_1]^{\omega_1}$. There is a countable set $A \subset Y$ such that $\{x_\gamma : \gamma \in A\}$ is dense in $\{x_\beta : \beta \in Y\}$. Applying (*) to A , we get an $\alpha < \omega_1$ satisfying the condition given there. Now choose $\beta \in Y \setminus A$ with $\beta \geq \alpha$. Since $\{x_\gamma : \gamma \in A\}$ is dense in $\{x_\beta : \beta \in Y\}$, x_β is in the closure of $\{x_\gamma : \gamma \in A\}$. As α is as in (*), there is a $\gamma \in A \subset Y$ so that $x_\gamma \subset x_\beta$, whence Y is not an antichain.

To fulfill condition (*), we must diagonalize against all countable subsets of ω_1 , and there are precisely \mathfrak{c} of these. But now the problem is that we must “catch” each countable subset of ω_1 at some *countable ordinal* α , and then we need to “respect” that countable set at all stages $\beta \geq \alpha$. Thus we need to be able to “trap” \mathfrak{c} subsets in ω_1 steps, and of course we can do this if $\mathfrak{c} = \omega_1$. Given CH, we can let $\langle A_\alpha : \alpha < \omega_1 \rangle$ enumerate all countable subsets of ω_1 in such a way that $\forall \alpha < \omega_1 [A_\alpha \subset \alpha]$. At a stage $\alpha < \omega_1$, we want to “respect” all the countable sets $\{A_\beta : \beta \leq \alpha\}$, which by assumption, are all subsets of α . We must do a sub-diagonalization against $\{x_\beta : \beta < \alpha\}$ and against $\{A_\beta : \beta \leq \alpha\}$ to produce a set $x_\alpha \in [\omega]^\omega$ satisfying:

1. $\omega \setminus x_\alpha$ is infinite
2. for each $\beta \leq \alpha$, if x_α is in the closure of $\{x_\gamma : \gamma \in A_\beta\}$, then $\exists \gamma \in A_\beta [x_\gamma \subset x_\alpha]$
3. $\forall \beta < \alpha [x_\alpha \not\subset x_\beta]$.

Since α is countable, it is easy to see that this sub-diagonalization can be performed, provided that $\forall \beta < \alpha [\omega \setminus x_\beta$ is infinite] (which we guarantee during the construction). In the argument sketched above, the \mathfrak{c} countable subsets of ω_1 can be dealt with in ω_1 steps because $\mathfrak{c} = \omega_1$. However, as we will see in the next section, it is not necessary to make the entire continuum small; all that is needed is a set of reals of size ω_1 “suitably capturing” the continuum.

We have seen that CH facilitates diagonalizations of length \mathfrak{c} by allowing the necessary sub-diagonalizations to be carried out and also by allowing some features of the continuum to be “captured” in a “small” number of steps. CH achieves both of these things by making the continuum as small as possible. But it is not necessary to ensure that all the sub-diagonalizations be of countable length in order to guarantee that

they can be done. There are appropriate assumptions guaranteeing that sub-diagonalizations of any uncountable length less than \mathfrak{c} can be performed. These are assumptions about cardinal invariants of the continuum, which we discuss in the next section. There are also assumptions about these cardinal invariants that provide a small set of reals capturing some combinatorial property of the entire continuum. In general, these assumptions tend to contradict those assumptions ensuring the feasibility of sub-diagonalizations of uncountable length, and hence cardinal invariants can be used to separate the two roles played by CH in diagonalizations of length \mathfrak{c} that become conflated when CH itself is assumed.

§2. Cardinal Invariants of the Continuum. A cardinal invariant of the continuum marks the place where a given type of diagonalization argument that works for any countable ordinal first fails; a cardinal invariant can be associated with each type of diagonalization argument. Moreover, there is always a set of size \mathfrak{c} for which these diagonalization arguments fail, so that every cardinal invariant lies between ω_1 and \mathfrak{c} (since the diagonalization always works for countable ordinals). We will illustrate the idea here with some examples, and we will introduce more of them as and when we need them for our discussion. A general survey of cardinal invariants may be found in [9] and [6]. We start with one of the most important types of diagonalization argument. Let us say that a family $\mathcal{A} \subset [\omega]^\omega$ has the *finite intersection property (FIP)* if for any finite $\{a_0, \dots, a_k\} \subset \mathcal{A}$, $|a_0 \cap \dots \cap a_k| = \omega$. Given a countable family $\mathcal{A} \subset [\omega]^\omega$ with the FIP, it is easy to diagonalize against it to produce a set $b \in [\omega]^\omega$ which is almost contained in all $a \in \mathcal{A}$. Indeed, if $\mathcal{A} = \{a_n : n \in \omega\}$, then simply let $b = \{k_0 < k_1 < \dots\}$, where k_n is the least element of $a_0 \cap \dots \cap a_n$ that is greater than k_{n-1} . Note that by assumption $a_0 \cap \dots \cap a_n$ is an infinite set. This is a diagonalization of countable length. Now, the cardinal invariant \mathfrak{p} , called the *pseudointersection number*, marks the first place where this diagonalization *cannot* be carried out. Notice there is always a set of size \mathfrak{c} for which this diagonalization fails (e.g. an ultrafilter).

DEFINITION 2.1.

$$\mathfrak{p} = \min \{|\mathcal{A}| : \mathcal{A} \subset [\omega]^\omega \text{ has FIP} \wedge \neg \exists b \in [\omega]^\omega \forall a \in \mathcal{A} [b \subset^* a]\}.$$

Another important type of diagonalization involves functions from ω to ω . Given f and g in ω^ω , we write $f <^* g$ to mean $\forall^\infty n \in \omega [f(n) < g(n)]$. We say that f *bounds* a family $\mathcal{A} \subset \omega^\omega$ if $\forall g \in \mathcal{A} [g <^* f]$. Again, an easy diagonalization of countable length shows that for every countable family $\mathcal{A} \subset \omega^\omega$, there is $f \in \omega^\omega$ that bounds it. Notice that the same diagonalization also produces a function (namely f) that is *not bounded* by any element of \mathcal{A} (which is weaker than bounding \mathcal{A}). The *bounding number* \mathfrak{b} is the first place where the diagonalization needed to produce

a function bounding a given family \mathcal{A} cannot be carried out, and the *dominating number* \mathfrak{d} is the first place where a function not bounded by any element of \mathcal{A} cannot be produced. Again notice that there is an \mathcal{A} of size \mathfrak{c} for which neither of these diagonalizations can be done (e.g. ω^ω).

DEFINITION 2.2.

$$\begin{aligned}\mathfrak{b} &= \min \{ |\mathcal{A}| : \mathcal{A} \subset \omega^\omega \wedge \mathcal{A} \text{ is unbounded} \}. \\ \mathfrak{d} &= \min \{ |\mathcal{A}| : \mathcal{A} \subset \omega^\omega \wedge \forall f \in \omega^\omega \exists g \in \mathcal{A} [f <^* g] \}.\end{aligned}$$

The failure of such diagonalization process implies the existence of a small set of reals which shares some combinatorial property of the whole continuum. We will illustrate this point with \mathfrak{b} and \mathfrak{d} . If \mathfrak{d} is “small”, say $\mathfrak{d} = \omega_1$, then there is a set $\mathcal{A} \subset \omega^\omega$ of size ω_1 , such that for every $f \in \omega^\omega$, there is $g \in \mathcal{A}$ such that $f <^* g$. Note just such an \mathcal{A} is simply a cofinal subset of the poset $(\omega^\omega, <^*)$. Thus $\mathfrak{d} = \omega_1$ is equivalent to the statement that the poset $(\omega^\omega, <^*)$ has a cofinal subset of size ω_1 . Much essential combinatorial information about a poset is contained in any of its cofinal subsets. Therefore, even if ω^ω is a large set, if \mathfrak{d} is “small”, then there is a “small” set capturing essential combinatorial information about the structure $(\omega^\omega, <^*)$. Similarly, if $\mathfrak{b} = \omega_1$, a different sort of combinatorial feature is captured by a set of reals of size ω_1 – i.e. there is a set of reals of size ω_1 that is unbounded in the poset $(\omega^\omega, <^*)$.

We observed in Section 1 that CH is used in diagonalizations of length continuum essentially for two related purposes: one is to ensure that the various sub-diagonalizations needed at each stage of the main diagonalization are all of countable length, and hence can be carried out; the other is to ensure the existence of a “small” set of reals capturing some essential combinatorial property of the entire continuum. We pointed out that CH achieves the first by achieving the second, and that it is desirable to separate them. Since cardinal invariants mark the places where a diagonalization which always works for countable sets first fails, and since the failure of such diagonalizations at small values points to the existence of small sets capturing some combinatorial essence of the continuum, CH can usually be replaced in these arguments with appropriate assumptions about cardinal invariants.

Notice that CH has two implication for cardinals invariants: it implies that every cardinal invariant is equal to \mathfrak{c} ; it also implies that every cardinal invariant is equal to ω_1 . Of course, under CH, these implication are not different; but in the absence of CH, they can be used to separate the two goals simultaneously achieved by CH. Observe that an assumption of the form $\mathfrak{x} = \mathfrak{c}$, where \mathfrak{x} is some cardinal invariant, allows the diagonalization process associated with \mathfrak{x} to proceed up to any length less than \mathfrak{c} , and hence can replace CH in diagonalizations of length \mathfrak{c} of the first kind. On the other hand, assumptions of the form $\mathfrak{x} = \omega_1$ provide a small set

of reals that capture some combinatorial feature of the continuum, and hence can be used for diagonalizations of length \mathfrak{c} of the second kind. This pattern obtains for the examples given in Section 1. Ketonen proved (see Theorem 4.4.5 of [7]) that if $\mathfrak{d} = \mathfrak{c}$, then there is a P-point. The following result of Todorćević [27] shows that the object constructed in the second example from Section 1 can be constructed as long as $\mathfrak{b} = \omega_1$ (it is easy to see that the hypothesis of the theorem below holds iff $\mathfrak{b} = \omega_1$).

THEOREM 2.3 (see Lemma 1.0. of [27]). *Let $L = \langle f_\alpha : \alpha < \omega_1 \rangle \subset \omega^\omega$ be such that*

1. $f_\alpha(n) \leq f_\alpha(n+1)$
2. $\forall \beta < \alpha < \omega_1 [f_\beta <^* f_\alpha]$
3. L is unbounded.

For each $\alpha < \omega_1$, put $x_\alpha = \{\langle n, m \rangle \in \omega \times \omega : m \leq f(n)\}$. Then the set $X = \langle x_\alpha : \alpha < \omega_1 \rangle \subset \omega \times \omega$ satisfies

- (a) *for each $A \in [\omega_1]^\omega$, there is an $\alpha < \omega_1$ so that for all $\beta \geq \alpha$, if x_β is in the closure of $\{x_\gamma : \gamma \in A\}$, then $\exists \gamma \in A [x_\gamma \subset x_\beta]$*
- (b) $\forall \beta < \alpha [x_\alpha \not\subset x_\beta]$.

When CH is replaced with assumptions about cardinal invariants of the continuum, the differences between the two types of diagonalizations of length \mathfrak{c} become apparent. For one thing, assumptions of the form $\mathfrak{x} = \mathfrak{c}$ are often inconsistent with assumptions of the form $\mathfrak{h} = \omega_1$ (unless, of course, $\mathfrak{c} = \omega_1$). But there is a far deeper and more fundamental distinction between the two. Natural forcing axioms such as Martin's Axiom (MA) and the Proper Forcing Axiom (PFA) imply that all cardinal invariants are equal to \mathfrak{c} . Therefore, all diagonalization of length \mathfrak{c} of the first kind can be carried out under natural forcing axioms like MA or PFA. On the other hand, MA (or more accurately, MA_{\aleph_1}) and PFA tend to *contradict* the very existence of the objects that are constructed in diagonalizations of length \mathfrak{c} of the second kind. For example, we have the following theorem of Kunen.

THEOREM 2.4 (see Lemma 1 of Baumgartner [8]). *Assume MA_{\aleph_1} . Let $X \subset [\omega]^\omega$ be well founded and uncountable. Then there exists an uncountable $A \subset X$ which is an antichain.*

This result shows that MA_{\aleph_1} kills off all objects of the kind constructed in the second example of Section 1, or given by Theorem 2.3 (in addition to making $\mathfrak{b} > \omega_1$). More generally, MA_{\aleph_1} implies that there are no first countable S or L spaces (Szentmiklóssy [26]). Other natural forcing axioms such the P-Ideal Dichotomy of Todorćević [28] also contradict the existence of such objects provided that $\mathfrak{p} > \omega_1$. Thus diagonalizations of the second kind really require some aspect of the continuum to be

“small” (to be ω_1). For diagonalizations of the first kind, all aspects of the continuum are allowed to be “big”.

In the rest of the this article we will focus on diagonalizations of length \mathfrak{c} of the first kind, particularly ones showing up in constructions of almost disjoint families of various sorts. We will present two such constructions that can be done without any additional hypotheses, even though they appear to require assumptions of the form $\mathfrak{r} = \mathfrak{c}$ at first sight. The two problems (and their solutions) share several formal similarities. Both constructions manage to avoid all extra assumptions by making use of non-trivial combinatorial characterizations of certain cardinal invariants. These two constructions are illustrative of a larger pattern: occasionally, some diagonalizations of the first kind can be done in ZFC by making clever use of combinatorial properties of well chosen cardinal invariants.

Before taking leave of diagonalizations of the second kind, we mention that it is also occasionally possible to do some of these in ZFC. The main technique here is that of minimal walks invented by Todorćević. This technique is useful for dealing with (among other things) problems that are “really about” the combinatorics of ω_1 . For each limit ordinal $\alpha < \omega_1$, one chooses a strictly increasing ω -sequence $C_\alpha \subset \alpha$ that is cofinal in α . One can then define various functions from ω_1 to ω using the fact that for any $\beta < \alpha < \omega_1$, $C_\alpha \cap \beta$ is a finite set. With the help of these functions it is possible to build using just ZFC combinatorial structures on ω_1 that may require a diagonalization of the second kind. A recent success of this technique is Moore’s construction of an L space on the basis of ZFC alone. For more details about the technique of minimal walks see Todorćević [29] and Moore [21].

§3. Almost Disjoint Families. In this section we introduce almost disjoint families. We say that two infinite subsets a and b of ω are *almost disjoint* or *a.d.* if $a \cap b$ is finite. We say that a family \mathcal{A} of infinite subsets of ω is *almost disjoint* or *a.d.* if its members are pairwise almost disjoint. Examples of a.d. families include $\{\{n : n \text{ is even}\}, \{n : n \text{ is odd}\}\}$ and $\{\{p^k : k \in \omega\} : p \text{ is a prime}\}$. Observe that the first of these is maximal in the sense that no a.d. family can properly contain it. However, in set theory, we are mainly interested in infinite examples that have this property. So we shall reserve the term *Maximal Almost Disjoint family*, or *MAD family* for an infinite a.d. family that is not properly contained in a larger a.d. family. And we will refer to a family as in the first example – i.e. a finite a.d. $\mathcal{A} \subset [\omega]^\omega$ such that $\bigcup \mathcal{A} \text{ }^* \supset \omega$ – as a *finite partition of* ω . Note that any infinite a.d. family, such as the second example above, can be extended to a MAD family using Zorn’s Lemma. It should be clear that ω can be replaced in this entire discussion by an arbitrary countable set X .

Now, any family of pairwise *disjoint* infinite subsets of ω must be countable. But it is an interesting combinatorial fact that every MAD family of infinite subsets of ω must be uncountable. And, of course, the argument is a diagonalization of length ω . Let $\mathcal{A} = \{a_n : n \in \omega\}$ be a countable a.d. family. We define a sequence of natural numbers $k_0 < k_1 < \dots$ by recursion as follows. k_0 is the least element of a_0 . Given k_n , we define k_{n+1} to be the least element of $a_{n+1} \setminus (a_0 \cup \dots \cup a_n)$ that is greater than k_n . This definition makes sense because a_{n+1} is almost disjoint from a_0, \dots, a_n , and so $a_{n+1} \setminus (a_0 \cup \dots \cup a_n)$ is an infinite set. Now, it is easy to see that the set $a = \{k_n : n \in \omega\}$ is an infinite set almost disjoint from all a_n . In the spirit of Section 2 we want to associate a cardinal number with this specific type of diagonalization, telling us the first place where it fails.

DEFINITION 3.1. $\mathfrak{a} = \min \{|\mathcal{A}| : \mathcal{A} \subset [\omega]^\omega \text{ is MAD in } [\omega]^\omega\}$

MAD families have been intensively studied in set theory. They have numerous applications in set theory as well as general topology. For example, the technique of almost disjoint coding has been used in forcing theory (see [19]) and MAD families are used in the construction of the Isbell-Mrówka space in topology (see [14]). Another connection with topology is the relation between almost disjoint refinements and \mathfrak{c} -points in the Stone-Ćech compactification of ω (see [4] and [1]). We will consider this last topic in greater detail soon.

There are several important variations on the notion of a MAD family of subsets of ω that will interest us. Two functions f and g in ω^ω are said to be *almost disjoint* or *a.d.* if they agree in only finitely many places. Such functions are sometimes also referred to as being eventually different. Identifying functions with their graphs – i.e. with certain subsets of $\omega \times \omega$ – we see that f and g are a.d. iff $|f \cap g| < \omega$. Similarly, we say that a family $\mathcal{A} \subset \omega^\omega$ is a.d. if its members are pairwise a.d., and finally we say that an a.d. family $\mathcal{A} \subset \omega^\omega$ is *MAD* if $\forall f \in \omega^\omega \exists h \in \mathcal{A} [|f \cap h| = \omega]$ (we do not need to stipulate that the family be infinite because no finite a.d. family of functions in ω^ω can be maximal). We will refer to an a.d. family of infinite subsets of a countable set X as an a.d. family in $[X]^\omega$, and to an a.d. family of functions as an a.d. family in ω^ω . Notice that any a.d. family $\mathcal{A} \subset \omega^\omega$ is also an a.d. family in $[\omega \times \omega]^\omega$, although it can *never* be a MAD family in $[\omega \times \omega]^\omega$ because every function is a.d. from all the vertical columns of $\omega \times \omega$. Once again, a countable diagonalization shows that no countable a.d. family $\mathcal{A} \subset \omega^\omega$ can be maximal. And, of course, analogously to \mathfrak{a} , associated with MAD families in ω^ω is yet another cardinal invariant.

DEFINITION 3.2. $\mathfrak{a}_\epsilon = \min \{|\mathcal{A}| : \mathcal{A} \subset \omega^\omega \text{ is MAD in } \omega^\omega\}$.

One of the two constructions of a.d. families we will present concerns almost disjointness in $[\omega]^\omega$ while the other is about ω^ω .

§4. Almost Disjoint Refinements. We deal here with the first of the two constructions of almost disjoint families we have alluded to above. The problem, or rather, group of related problems concerns when we can find an almost disjoint refinement of a family $\mathcal{C} \subset [\omega]^\omega$. Given $\mathcal{C} \subset [\omega]^\omega$, we say that a family $\mathcal{A} = \{a_c : c \in \mathcal{C}\} \subset [\omega]^\omega$ is an *almost disjoint refinement (ADR)* of \mathcal{C} if

1. $\forall c \in \mathcal{C} [a_c \subset c]$
2. $\forall c_0, c_1 \in \mathcal{C} [c_0 \neq c_1 \implies |a_{c_0} \cap a_{c_1}| < \omega]$.

Not every $\mathcal{C} \subset [\omega]^\omega$ can have an ADR. Recall that we say that a family $\mathcal{I} \subset \mathcal{P}(\omega)$ is an *ideal* on ω if \mathcal{I} is closed under subsets and pairwise unions, and also contains all finite subsets of ω . We say that a family $\mathcal{X} \subset [\omega]^\omega$ is *dense* if $\forall x \in [\omega]^\omega \exists y \in [x]^\omega [y \in \mathcal{X}]$, and we say that \mathcal{X} is *open* if $\forall x \in \mathcal{X} \forall y \in [\omega]^\omega [y \subset^* x \implies y \in \mathcal{X}]$. Note that any ideal on ω is automatically open. Now suppose \mathcal{C} has an ADR, $\mathcal{A} = \{a_c : c \in \mathcal{C}\}$. We claim that there is a dense ideal \mathcal{I} such that $\mathcal{I} \cap \mathcal{C} = 0$. To see this, let \mathcal{I} be an ideal on ω maximal with respect to the condition that $\mathcal{I} \cap \mathcal{A} = 0$. Notice that if an ideal \mathcal{I} fails to intersect \mathcal{A} , then it doesn't intersect \mathcal{C} either. Now, we will argue that this \mathcal{I} is dense. If not, then there is a set $x \in [\omega]^\omega$ such that no $y \in [x]^\omega$ is in \mathcal{I} . Moreover, by the maximality of \mathcal{I} , no such y can be added to \mathcal{I} without violating the requirement that $\mathcal{I} \cap \mathcal{A} = 0$. Therefore, for every $y \in [x]^\omega$, there is an $a \in \mathcal{I}$ and $c \in \mathcal{C}$ such that $a_c \subset y \cup a$. In particular, there is $a_0 \in \mathcal{I}$ and $c_0 \in \mathcal{C}$ such that $a_{c_0} \subset x \cup a_0$. Since $a_{c_0} \notin \mathcal{I}$, $a_{c_0} \setminus a_0$ is an infinite subset of x . Choose $z \in [a_{c_0} \setminus a_0]^\omega$ with $\bar{z} = (a_{c_0} \setminus a_0) \setminus z$ also infinite. Note that since \bar{z} is an infinite subset of x , $\bar{z} \notin \mathcal{I}$. Next, since $z \in [x]^\omega$, we can find $a_1 \in \mathcal{I}$ and $c_1 \in \mathcal{C}$ such that $a_{c_1} \subset z \cup a_1$. Once again, $a_{c_1} \setminus a_1$ is an infinite subset of $z \subset a_{c_0}$. So a_{c_0} and a_{c_1} have infinite intersection, whence $c_0 = c_1$. Now, we have $a_{c_0} \subset z \cup a_1$. But now, $\bar{z} \subset a_1$, contradicting our observation that $\bar{z} \notin \mathcal{I}$.

How about the converse? Suppose $\mathcal{C} \subset [\omega]^\omega$ is a family so that there is a dense ideal \mathcal{I} satisfying $\mathcal{C} \cap \mathcal{I} = 0$. Does \mathcal{C} have an ADR? Firstly, note that this is the same as asking whether $\mathcal{P}(\omega) \setminus \mathcal{I}$ has an ADR whenever $\mathcal{I} \subset \mathcal{P}(\omega)$ is a dense ideal. Let us consider what such a construction involves. Fix a dense ideal \mathcal{I} , and put $\mathcal{C} = \mathcal{P}(\omega) \setminus \mathcal{I}$. Write $\mathcal{C} = \{c_\alpha : \alpha < \mathfrak{c}\}$. At each stage $\alpha < \mathfrak{c}$, we are given $\{a_\beta : \beta < \alpha\}$, an ADR for $\{c_\beta : \beta < \alpha\}$. We need to do a sub-diagonalization against the set $\{a_\beta : \beta < \alpha\}$ to find an $a_\alpha \subset c_\alpha$ that is almost disjoint from all a_β . We must have chosen the a_β carefully for this to be possible. Given an a.d. family $\mathcal{A} \subset [\omega]^\omega$, let $\mathcal{I}(\mathcal{A})$ denote the ideal on ω generated by \mathcal{A} . For the sub-diagonalization to have any hope of succeeding, c_α must not be in $\mathcal{I}(\{a_\beta : \beta < \alpha\})$, for otherwise any set a.d. from $\{a_\beta : \beta < \alpha\}$ will also be a.d. from c_α . Hence we must promise that at any stage α , $\mathcal{I}(\{a_\beta : \beta < \alpha\}) \cap \{c_\gamma : \gamma \geq \alpha\} = 0$. Of course this will not be a problem if we choose each $a_\beta \in \mathcal{I}$, because

then $\mathcal{I}(\{a_\beta : \beta < \alpha\}) \subset \mathcal{I}$, which we know is disjoint from \mathcal{C} . It is here that the density of \mathcal{I} comes into play. So assume that at stage α we have chosen $\{a_\beta : \beta < \alpha\} \subset \mathcal{I}$. Now, put $\mathcal{A}_\alpha = \{a_\beta : \beta < \alpha\}$ and $\mathcal{A}_\alpha \cap c_\alpha = \{a_\beta \cap c_\alpha : \beta < \alpha \wedge |a_\beta \cap c_\alpha| = \omega\}$. Notice that $\mathcal{A}_\alpha \cap c_\alpha$ is an almost disjoint family in $[c_\alpha]^\omega$, and that it is sufficient to find an $a_\alpha \in [c_\alpha]^\omega$ that is almost disjoint from all the members of $\mathcal{A}_\alpha \cap c_\alpha$. This requires $\mathcal{A}_\alpha \cap c_\alpha$ to be neither a MAD family in $[c_\alpha]^\omega$ nor a finite partition of c_α . Since $c_\alpha \notin \mathcal{I}(\mathcal{A}_\alpha)$, we know that $\mathcal{A}_\alpha \cap c_\alpha$ is not a finite partition of c_α . But, *a priori* there is no reason why it could not be a MAD family in $[c_\alpha]^\omega$. Notice however that $\mathcal{A}_\alpha \cap c_\alpha$ has fewer than \mathfrak{c} members. Therefore, if there are no “small” MAD families in $[\omega]^\omega$ – i.e. if $\mathfrak{a} = \mathfrak{c}$ – then $\mathcal{A}_\alpha \cap c_\alpha$ cannot be a MAD family in $[c_\alpha]^\omega$. So if $\mathfrak{a} = \mathfrak{c}$, there is a $b \in [c_\alpha]^\omega$ which is a.d. from the things in $\mathcal{A}_\alpha \cap c_\alpha$. Now, since \mathcal{I} is a dense ideal there is an $a_\alpha \in [b]^\omega$ which is in \mathcal{I} . Clearly, this a_α is also a.d. from the things in $\mathcal{A}_\alpha \cap c_\alpha$, and now the construction can proceed. Thus we have shown that if $\mathfrak{a} = \mathfrak{c}$, then every set of the form $\mathcal{P}(\omega) \setminus \mathcal{I}$, where \mathcal{I} is dense ideal, has an ADR.

Can this construction be done in ZFC? This question remains open in full generality, even though significant progress has recently been made by Shelah [25]. It is known to be equivalent to the problem of whether there is a completely separable MAD family $\mathcal{A} \subset \mathcal{I}$ for every dense ideal \mathcal{I} .

DEFINITION 4.1. A MAD family $\mathcal{A} \subset [\omega]^\omega$ is said to be *completely separable* if for any $b \in \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$, there is an $a \in \mathcal{A}$ with $a \subset b$.

It is not too hard to see that if \mathcal{A} is any MAD family, then for each $b \in \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$, there is an almost disjoint family of size continuum $\{c_\alpha : \alpha < \mathfrak{c}\} \subset [b]^\omega \cap (\mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A}))$. Using this fact and using some properties of the base matrix, a combinatorial object exposed below, it is possible to show that $\mathcal{P}(\omega) \setminus \mathcal{I}$ has an ADR for every dense ideal \mathcal{I} iff for every dense ideal \mathcal{I} , there is a completely separable MAD family $\mathcal{A} \subset \mathcal{I}$ (we are *not* asserting that these are equivalent for a given \mathcal{I}). We skip the proof of this as we will not need it in what follows. The proof may be found in [3].

Even though the ZFC existence of ADRs is an open question in general, Balcar and Vojtáš established it in 1980 for several specific cases by using a deep combinatorial property of the cardinal invariant \mathfrak{h} . Suppose $\{\mathcal{X}_n : n \in \omega\} \subset [\omega]^\omega$ is a countable collection of dense open sets. Suppose $b \in [\omega]^\omega$. As \mathcal{X}_0 is dense we can find $a_0 \in [b]^\omega \cap \mathcal{X}_0$, and as \mathcal{X}_1 is dense there is $a_1 \in [a_0]^\omega \cap \mathcal{X}_1$. Continuing in this way we get a sequence $b \supset a_0 \supset a_1 \supset \dots$, where $a_i \in \mathcal{X}_i$. We can find a $c \in [b]^\omega$ such that $\forall i \in \omega [c \subset^* a_i]$. As \mathcal{X}_i is open, we have $c \in \mathcal{X}_i$, and we conclude that the intersection of countably many dense open sets is dense. \mathfrak{h} marks the first place where this diagonalization cannot be carried out.

DEFINITION 4.2.

$$\mathfrak{h} = \min \{ |\mathcal{F}| : \forall \mathcal{X} \in \mathcal{F} [\mathcal{X} \subset [\omega]^\omega \text{ is dense open}] \wedge \bigcap \mathcal{F} \text{ is not dense} \}.$$

The definition would not change if we replaced “ $\bigcap \mathcal{F}$ is not dense” by “ $\bigcap \mathcal{F} = 0$ ”. An alternative way to define \mathfrak{h} is as the least κ such that the forcing notion $\mathcal{P}(\omega)/[\omega]^{<\omega}$ adds a new function from κ into the ground model \mathbf{V} .

The following deep combinatorial characterization of \mathfrak{h} was proved by Balcar, Pelant and Simon [2]. The tree \mathcal{T} whose existence is proved in the following theorem is often called a *base tree* or a *base matrix*. Its proof was the original motivation for the introduction of the cardinal \mathfrak{h} , which stands for the height of the base tree.

THEOREM 4.3. *There is a family $\mathcal{T} \subset [\omega]^\omega$ satisfying the following properties:*

1. $\langle \mathcal{T}, * \supset \rangle$ is a tree of height \mathfrak{h}
2. each level of \mathcal{T} is a MAD family in $[\omega]^\omega$ except the root, which is ω
3. each node has \mathfrak{c} immediate successors
4. \mathcal{T} is a dense family in $[\omega]^\omega$.

Moreover, \mathfrak{h} is the least height of any such tree.

As an aside, we remark that since \mathcal{T} is dense in $[\omega]^\omega$, forcing with $\mathcal{P}(\omega)/[\omega]^{<\omega}$ is equivalent to forcing with \mathcal{T} . Because of (3), forcing with \mathcal{T} adjoins an onto map from \mathfrak{h} to \mathfrak{c} . Moreover, forcing with $\mathcal{P}(\omega)/[\omega]^{<\omega}$ does not add a new function from any $\alpha < \mathfrak{h}$ into the ground model \mathbf{V} . It follows that forcing with $\mathcal{P}(\omega)/[\omega]^{<\omega}$ always makes $\mathfrak{c}^{\mathbf{V}[G]} = \mathfrak{h}^{\mathbf{V}}$.

We will illustrate how this theorem can be used to get ADRs by proving a simple instance of a result of Balcar and Vojtáš. Their major result was the solution to a problem of Pierce and Hindman that had been open for a long time. Notice that one example of a dense ideal is a maximal one. If \mathcal{I} is a maximal ideal, then $\mathcal{P}(\omega) \setminus \mathcal{I}$ is simply an ultrafilter. So we can ask does every ultrafilter have an ADR? This question was posed by Pierce [22] and Hindman [16] in the context of \mathfrak{c} -points in $\beta\omega \setminus \omega$, the Stone-Ćech remainder of ω . A point p in a topological space is called a *\mathfrak{c} -point* if there is a family of \mathfrak{c} pairwise disjoint open sets each of which has p in its closure. Pierce and Hindman asked if each point $\mathcal{U} \in \beta\omega \setminus \omega$ is a \mathfrak{c} -point. It is easy to see that this is the same as asking if every $\mathcal{U} \in \beta\omega \setminus \omega$ has an ADR. Balcar and Vojtáš [4] provided a positive answer in ZFC.

THEOREM 4.4. *Every non-principal ultrafilter has an ADR.*

We don't prove this here. But we illustrate some of the ideas involved by showing in ZFC that the complement of the following dense ideal defined on $\omega \times \omega$ has a ADR. This was also done in the paper Balcar and Vojtáš [4].

DEFINITION 4.5. Let $E \subset \omega \times \omega$. We define $E(n) = \{m \in \omega : \langle n, m \rangle \in E\}$, and $\text{dom}(E) = \{n \in \omega : E(n) \neq \emptyset\}$. We also define \mathcal{I}_0 as

$$\mathcal{I}_0 = \{E \subset \omega \times \omega : \exists k \forall^\infty n \in \omega [|E(n)| \leq k]\}.$$

Lastly, we define $\mathcal{E}_0 = \mathcal{P}(\omega \times \omega) \setminus \mathcal{I}_0$.

It is easily seen that \mathcal{I}_0 is a dense ideal. We get an ADR for \mathcal{E}_0 just in ZFC. We say that a set $F \in [\omega \times \omega]^\omega$ is *fast* if, letting $\{n_0 < n_1 < \dots\}$ enumerate $\text{dom}(F)$, we have that for each i , $F(n_i)$ is a non-empty finite set, and $|F(n_{i+1})| > |F(n_i)|$. It is straightforward to see that each $E \in \mathcal{E}_0$ contains a fast $F \subset E$. Let $\{E_\alpha : \alpha < \mathfrak{c}\}$ enumerate \mathcal{E}_0 , and for each $\alpha < \mathfrak{c}$, choose a fast $F_\alpha \subset E_\alpha$. Clearly, it is enough to produce an ADR for $\{F_\alpha : \alpha < \mathfrak{c}\}$. Now, let us say that an infinite $p \in [\omega \times \omega]^\omega$ is an *infinite partial function* if $\forall n \in \omega [|p(n)| \leq 1]$. We build an almost disjoint sequence of infinite partial functions $\{p_\alpha : \alpha < \mathfrak{c}\}$ such that $p_\alpha \subset F_\alpha$. The trick is to ensure that for each $\alpha < \mathfrak{c}$, there is a $t_\alpha \in \mathcal{T}$ such that $\text{dom}(p_\alpha) \subset t_\alpha$. Fix $\alpha < \mathfrak{c}$, and assume $\{p_\beta : \beta < \alpha\}$, and $\{t_\beta : \beta < \alpha\} \subset \mathcal{T}$ have been chosen so that $\text{dom}(p_\beta) \subset t_\beta$, and so that $\forall \beta < \gamma < \alpha [t_\beta \neq t_\gamma]$. Now, because of clauses (3) and (4) of Theorem 4.3, there is a level $\delta < \mathfrak{h}$ of \mathcal{T} so that $X = \{t \in \mathcal{T}_\delta : t \subset^* \text{dom}(F_\alpha)\}$ has size \mathfrak{c} , where \mathcal{T}_δ is the δ^{th} level of \mathcal{T} . Since $\alpha < \mathfrak{c}$, and since \mathcal{T} is a tree, we can choose $t_\alpha \in X$ satisfying $\neg \exists \beta < \alpha [t_\beta \subset^* t_\alpha]$ (simply choose a $t \in X$ which is not a projection of some t_β to \mathcal{T}_δ). Now, since each level of \mathcal{T} is almost disjoint, observe that $|t_\beta \cap t_\alpha| < \omega$, unless $t_\alpha \subset^* t_\beta$. Therefore, unless $t_\alpha \subset^* t_\beta$, p_β is almost disjoint from $F_\alpha \upharpoonright t_\alpha = \{\langle n, m \rangle \in F_\alpha : n \in t_\alpha\}$. So we just need to deal with $\mathcal{A} = \{p_\beta : t_\alpha \subset^* t_\beta \wedge \beta < \alpha\}$. But since $\delta < \mathfrak{h}$, and since the t_β are all different, $|\mathcal{A}| < \mathfrak{h}$. Now, $\mathcal{A} \cap (F_\alpha \upharpoonright t_\alpha) = \{p_\beta \cap (F_\alpha \upharpoonright t_\alpha) : p_\beta \in \mathcal{A} \wedge |p_\beta \cap (F_\alpha \upharpoonright t_\alpha)| = \omega\}$ is an almost disjoint family in $[F_\alpha \upharpoonright t_\alpha]^\omega$. Moreover, it consists entirely of infinite partial functions. Since $F_\alpha \upharpoonright t_\alpha$ is fast, it is not a finite partition of $F_\alpha \upharpoonright t_\alpha$. But the following lemma tells us that \mathfrak{h} is “small”.

LEMMA 4.6. $\mathfrak{h} \leq \mathfrak{b} \leq \mathfrak{a}$.

Therefore, $|\mathcal{A} \cap (F_\alpha \upharpoonright t_\alpha)| < \mathfrak{a}$, and so it is not an infinite MAD family on $[F_\alpha \upharpoonright t_\alpha]^\omega$. So we can find a $x \in [F_\alpha \upharpoonright t_\alpha]^\omega$ which is almost disjoint from all p_β . As $F_\alpha(n)$ is finite for all n , we can find an infinite partial function $p_\alpha \subset x$, which finishes the construction.

§5. Van Douwen Families. In this section we discuss the second of the two ZFC constructions of almost disjoint families we have been alluding to. This one concerns MAD families in ω^ω . It answers an old question of Van Douwen. Van Douwen asked whether there is a MAD family of functions $\mathcal{A} \subset \omega^\omega$ that is also maximal with respect to infinite partial functions. Let us call such MAD families *Van Douwen*. Recall

from Section 4 that $p \in [\omega \times \omega]^\omega$ is an infinite partial function if for each n , $p(n) = \{m \in \omega : \langle n, m \rangle \in p\}$ has at most one element.

DEFINITION 5.1. An a.d. family $\mathcal{A} \subset \omega^\omega$ is said to be *Van Douwen* if for any infinite partial function p there is $h \in \mathcal{A}$ such that $|h \cap p| = \omega$.

There are several equivalent formulations of Van Douwen's question and it is instructive to consider some of them. Recall from Section 3 that any a.d. family of functions $\mathcal{A} \subset \omega^\omega$ is an a.d. family of sets in $[\omega \times \omega]^\omega$, but is never a MAD family in $[\omega \times \omega]^\omega$ because the vertical columns are almost disjoint from any function. Van Douwen MAD families are those a.d. families of functions that have nothing else preventing them from being MAD in $[\omega \times \omega]^\omega$. In other words an a.d. $\mathcal{A} \subset \omega^\omega$ is Van Douwen iff $\mathcal{A} \cup \{c_n : n \in \omega\}$ is MAD in $[\omega \times \omega]^\omega$, where c_n is the n^{th} vertical column of $\omega \times \omega$ – that is, $c_n = \{\langle n, m \rangle : m \in \omega\}$. To see this, suppose $\mathcal{A} \subset \omega^\omega$ is Van Douwen. If $x \in [\omega \times \omega]^\omega$ is almost disjoint from $\{c_n : n \in \omega\}$, then there is an infinite partial function $p \subset x$, whence there is $f \in \mathcal{A}$ such that $|f \cap x| = \omega$. Therefore, $\mathcal{A} \cup \{c_n : n \in \omega\}$ is MAD in $[\omega \times \omega]^\omega$. On the other hand, suppose $\mathcal{A} \subset \omega^\omega$ is an a.d. family such that $\mathcal{A} \cup \{c_n : n \in \omega\}$ is MAD in $[\omega \times \omega]^\omega$. Let p be an infinite partial function. p is a.d. from each c_n , and so there is $h \in \mathcal{A}$ so that $|p \cap h| = \omega$, whence \mathcal{A} is Van Douwen.

Another formulation is to ask whether there is an a.d. family $\mathcal{A} \subset \omega^\omega$ which is “everywhere maximal” in the following sense. Given an a.d. family $\mathcal{A} \subset \omega^\omega$ and a set $X \in [\omega]^\omega$, we can consider the restriction of \mathcal{A} to X , $\mathcal{A} \upharpoonright X = \{h \upharpoonright X : h \in \mathcal{A}\}$. This is an a.d. family of functions in ω^X . It is easily seen that \mathcal{A} is Van Douwen MAD iff all its restrictions are maximal – that is, $\mathcal{A} \upharpoonright X$ is MAD in ω^X for all $X \in [\omega]^\omega$.

There are several motivations for considering such families. One natural property of interest of maximal objects (of any sort) is preservation of their maximality in forcing extensions.

DEFINITION 5.2. Let \mathbb{P} be a notion of forcing and let \mathcal{A} be a MAD family (either of sets or of functions). We will say that \mathcal{A} is *\mathbb{P} -indestructible* if $\Vdash_{\mathbb{P}} \mathcal{A}$ is MAD.

Obviously, if \mathbb{P} is a forcing notion that does not add a real, then every MAD \mathcal{A} is \mathbb{P} -indestructible. The most basic example of a \mathbb{P} that adds a real is Cohen forcing (i.e. $\text{Fn}(\omega, 2)$). When is a MAD \mathcal{A} Cohen indestructible? It turns out Cohen indestructibility is closely related to the combinatorial notion of a strongly MAD family. Brendle and Yatabe [11] have provided combinatorial characterizations of the property of being a \mathbb{P} -indestructible MAD family of sets in $[\omega]^\omega$ for some standard posets \mathbb{P} .

DEFINITION 5.3. An a.d. family of sets $\mathcal{A} \subset [\omega]^\omega$ is *strongly MAD* if for each countable collection of sets $\{a_n : n \in \omega\} \subset \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$, where

$\mathcal{I}(\mathcal{A})$ is the ideal on ω generated by \mathcal{A} , $\exists c \in \mathcal{A} \forall n \in \omega [|c \cap a_n| = \omega]$. An a.d. family of functions $\mathcal{A} \subset \omega^\omega$ is *strongly MAD* if for each countable collection of functions $\{f_n : n \in \omega\} \subset \omega^\omega \cap (\mathcal{P}(\omega \times \omega) \setminus \mathcal{I}(\mathcal{A}))$, where $\mathcal{I}(\mathcal{A})$ is the ideal on $\omega \times \omega$ generated by \mathcal{A} , $\exists h \in \mathcal{A} \forall n \in \omega [|h \cap f_n| = \omega]$.

The requirement that the collection of the a_n , and the collection of the f_n , miss $\mathcal{I}(\mathcal{A})$ is essential, for no element of \mathcal{A} can have infinite intersection with two distinct members of \mathcal{A} .

For MAD families in $[\omega]^\omega$, strong MADness is “almost equivalent” to Cohen indestructibility. More precisely, a MAD family of sets $\mathcal{A} \subset [\omega]^\omega$ is Cohen indestructible iff there is a set $X \in \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$ such that $\mathcal{A} \cap X$ is strongly MAD in $[X]^\omega$. For MAD families of functions in ω^ω , Cohen indestructibility is weaker than strong MADness. But for both types of MAD families strong MADness implies Cohen indestructibility. An a.d. family $\mathcal{A} \subset \omega^\omega$ is strongly MAD iff $\mathcal{A} \cup \{c_n : n \in \omega\}$ is strongly MAD in $[\omega \times \omega]^\omega$. Therefore, a strongly MAD $\mathcal{A} \subset \omega^\omega$ not only remains maximal in ω^ω after the addition of a Cohen real, but even $\mathcal{A} \cup \{c_n : n \in \omega\}$ remains maximal with respect to $[\omega \times \omega]^\omega$. In particular, a strongly MAD $\mathcal{A} \subset \omega^\omega$ is Van Douwen, and indeed, it is Cohen indestructibly Van Douwen.

Why must a strongly MAD $\mathcal{A} \subset \omega^\omega$ be Van Douwen? In fact, an a.d. family $\mathcal{A} \subset \omega^\omega$ must be Van Douwen even if it has the property that for every $\{f_0, f_1\} \subset \omega^\omega \cap (\mathcal{P}(\omega \times \omega) \setminus \mathcal{I}(\mathcal{A}))$, $\exists h \in \mathcal{A} \forall i \in 2 [|h \cap f_i| = \omega]$. Indeed, suppose for a contradiction that such an \mathcal{A} is not Van Douwen. Let p be an infinite partial function that is a.d. from \mathcal{A} . Choose $h_0, h_1 \in \mathcal{A}$ so that $h_0 \neq h_1$ (any such \mathcal{A} must be uncountable; so this is possible). Put $a = \omega \setminus \text{dom}(p)$, and for each $i \in 2$, set $f_i = p \cup (h_i \upharpoonright a)$. Since p was supposed to be almost disjoint from \mathcal{A} , it follows that $\{f_0, f_1\} \subset \omega^\omega \cap (\mathcal{P}(\omega \times \omega) \setminus \mathcal{I}(\mathcal{A}))$. Choose $h \in \mathcal{A}$ so that $|h \cap f_0| = |h \cap f_1| = \omega$. Since h is a.d. from p , we must have that $|h \cap h_0| = |h \cap h_1| = \omega$. But then $h = h_0$, and $h = h_1$, contradicting our choice of h_0 and h_1 . It is not known whether it is possible to construct a strongly MAD family (of either sort) just in ZFC. They can be constructed if $\mathfrak{b} = \mathfrak{c}$. However, the following natural strengthening of the notion of a strongly MAD family of functions was introduced by Kastermans [18].

DEFINITION 5.4. Let \mathcal{A} be an a.d. family and put $\kappa = |\mathcal{A}|$. We say that \mathcal{A} is *very MAD* if for every cardinal $\lambda < \kappa$ and for every collection $\{f_\alpha : \alpha < \lambda\} \subset \omega^\omega \cap (\mathcal{P}(\omega \times \omega) \setminus \mathcal{I}(\mathcal{A}))$, $\exists h \in \mathcal{A}$ such that $\forall \alpha < \lambda [|h \cap f_\alpha| = \omega]$.

Kastermans [18] showed that very MAD families exist if $\mathfrak{p} = \mathfrak{c}$, and he asked if their existence be proved in ZFC. Regarding this it was conjectured by Brendle that if $\text{cov}(\mathcal{M}) < \mathfrak{a}_\mathfrak{c}$, then there are no very MAD families, and moreover, he verified that there are no very MAD families in several natural models of the statement $\text{cov}(\mathcal{M}) < \mathfrak{a}_\mathfrak{c}$, such as the

Laver model, and the random real model. Here $\text{cov}(\mathcal{M})$ is yet another cardinal invariant; but unlike the ones considered so far, its rationale lies with some topological properties of the reals. Recall that a set $A \subset X$ of some topological space X is *nowhere dense* if the closure of A has empty interior. $A \subset X$ is *meager* if it is a countable union of nowhere dense sets. For the Baire space ω^ω , we know from the Baire category theorem that countably many meager sets cannot cover ω^ω , and we also have, by definition, that any countable set $A \subset \omega^\omega$ must be meager. Thus, we are led to the following natural cardinal invariants.

DEFINITION 5.5.

$$\text{cov}(\mathcal{M}) = \min \{ |\mathcal{F}| : \forall A \in \mathcal{F} [A \subset \omega^\omega \text{ is meager in } \omega^\omega] \wedge \bigcup \mathcal{F} = \omega^\omega \}.$$

$$\text{non}(\mathcal{M}) = \min \{ |A| : A \subset \omega^\omega \text{ is not meager in } \omega^\omega \}.$$

In [23], we proved Brendle’s conjecture:

THEOREM 5.6. *If \mathcal{A} is a very MAD family, then $|\mathcal{A}| \leq \text{cov}(\mathcal{M})$. In particular, if $\text{cov}(\mathcal{M}) < \mathfrak{a}_\mathfrak{c}$, then there are no very MAD families.*

Thus the notion of a Van Douwen family is the weakest in a natural progression of notions of MADness of increasing strength. Whereas we know that MAD families of the strongest variety cannot be built in ZFC, it is unknown if those of the middle variety can, while in [24] we built a Van Douwen family of size \mathfrak{c} in ZFC showing that ones of the weakest kind can.

Van Douwen’s question dates to the 1980s. It occurs as problem 4.2 in Miller’s problem list [20]. In 1999 Zhang [31] obtained some partial results on this problem. He proved that Van Douwen MAD families of various sizes exist in certain forcing extensions. Let us consider what constructing such a family involves. Let $\langle p_\alpha : \alpha < \mathfrak{c} \rangle$ enumerate all infinite partial functions. At a stage $\alpha < \mathfrak{c}$, we are given an almost disjoint family of functions $\mathcal{A}_\alpha \subset \omega^\omega$ with $|\mathcal{A}_\alpha| < \mathfrak{c}$ and an infinite partial function p_α . If p_α is not almost disjoint from \mathcal{A}_α , then there is nothing to do. If it is, then we must perform a sub-diagonalization against the functions in \mathcal{A}_α to produce a $h \in \omega^\omega$ such that $\forall f \in \mathcal{A}_\alpha [|h \cap f| < \omega]$ and $|h \cap p_\alpha| = \omega$. Of course, the problem is that \mathcal{A}_α may already be maximal in ω^ω . As $|\mathcal{A}_\alpha| < \mathfrak{c}$, this will not be a problem if there are no “small” MAD families in ω^ω – i.e. if $\mathfrak{a}_\mathfrak{c} = \mathfrak{c}$. If $\mathfrak{a}_\mathfrak{c} = \mathfrak{c}$, then there is a $g \in \omega^\omega$ such that $\forall f \in \mathcal{A}_\alpha [|g \cap f| < \omega]$. Now, put $a = \omega \setminus \text{dom}(p_\alpha)$, and simply take h to be $p_\alpha \cup (g \upharpoonright a)$. Note that h is almost disjoint from \mathcal{A}_α because p_α was assumed to be. Thus we conclude that if $\mathfrak{a}_\mathfrak{c} = \mathfrak{c}$, then there is a Van Douwen MAD family. This situation is formally analogous to the one for ADRs: both diagonalizations succeed *provided* that we do not end up with a MAD family “too soon”.

Continuing the analogy with ADRs, a Van Douwen MAD family can be constructed in ZFC using a deep combinatorial characterization of an appropriate cardinal invariant. Here the right invariant turns out to be $\text{non}(\mathcal{M})$ introduced in Definition 5.5. The analogue of Theorem 4.3 here is the following combinatorial characterization of $\text{non}(\mathcal{M})$ proved by Bartoszyński [5].

DEFINITION 5.7. Let $h \in \omega^\omega$. An *h-slalom* is a function $S : \omega \rightarrow [\omega]^{<\omega}$ such that for all $n \in \omega$, $|S(n)| \leq h(n)$.

THEOREM 5.8 (Bartoszyński). *Let κ be an infinite cardinal. The following are equivalent:*

1. *Every set of reals of size less than κ is meager.*
2. *For every family $F \subset \omega^\omega$ with $|F| < \kappa$, there is an infinite partial function p such that $\forall f \in F [|p \cap f| < \omega]$.*
3. *For every h and for every family of h -slaloms F with $|F| < \kappa$, there is a $g \in \omega^\omega$ such that $\forall S \in F \forall^\infty n \in \omega [g(n) \notin S(n)]$.*

Crucial to the construction of a Van Douwen family is (3) in the above theorem. The first step is to reformulate Van Douwen's question as follows. Fix an a.d. family $\mathcal{A} \subset \omega^\omega$ and $f \in \omega^\omega$. Recall that $\mathcal{A} \cap f = \{h \cap f : h \in \mathcal{A} \wedge |h \cap f| = \omega\}$ is an a.d. family of sets in $[f]^\omega$. It may or may be maximal in $[f]^\omega$. If it is maximal, it may either be a finite partition of f , or it may be a MAD family in $[f]^\omega$. It is clear that $\mathcal{A} \subset \omega^\omega$ is Van Douwen MAD iff for every $f \in \omega^\omega$, $\mathcal{A} \cap f$ is either a finite partition of f or is MAD in $[f]^\omega$.

DEFINITION 5.9. Let $\mathcal{A} \subset \omega^\omega$ be an a.d. family. The *trace of \mathcal{A}* , written $\text{tr}(\mathcal{A})$, is

$$\{f \in \omega^\omega : \text{either } \mathcal{A} \cap f \text{ is a finite partition of } f \text{ or is MAD in } [f]^\omega\}.$$

Thus, Van Douwen families are those a.d. families in ω^ω whose trace is as large as possible. But it is not necessary to ensure that $\text{tr}(\mathcal{A})$ is ω^ω ; it is enough if it is non-meager in ω^ω . Let p be any infinite partial function. It is easy to see that $\{f \in \omega^\omega : |p \cap f| < \omega\}$ is meager in ω^ω . Therefore, if $F \subset \omega^\omega$ is a non-meager set, then for any infinite partial function p , $\exists f \in F [|p \cap f| = \omega]$. In particular, we can fix such an F with $|F| = \text{non}(\mathcal{M})$. It is sufficient for $F \subset \text{tr}(\mathcal{A})$ for \mathcal{A} to be Van Douwen.

Let $F = \{f_\alpha : \alpha < \text{non}(\mathcal{M})\} \subset \omega^\omega$ be a non-meager set. We need to construct an a.d. $\mathcal{A} \subset \omega^\omega$ such that $F \subset \text{tr}(\mathcal{A})$. So now we only have $\text{non}(\mathcal{M})$ many requirements to meet. Of course, $\text{non}(\mathcal{M})$ may be \mathfrak{c} , so this may not seem like much of an improvement. But as $\text{non}(\mathcal{M}) \leq \mathfrak{a}_e$, if $\text{non}(\mathcal{M}) = \mathfrak{c}$, then $\mathfrak{a}_e = \mathfrak{c}$ as well, and we already know how to construct a Van Douwen family in this case. So in the situation that interests us, $\text{non}(\mathcal{M}) < \mathfrak{c}$. However, the *real* point is that, we can now use Theorem

5.8 because at each stage, we would have “used up fewer than $\text{non}(\mathcal{M})$ things”.

So instead of building a family of size \mathfrak{c} in \mathfrak{c} steps, we construct it in $\text{non}(\mathcal{M})$ many steps, and instead of adding one object to the family at each stage, we add \mathfrak{c} . To start, how do we make sure that $f_0 \in \text{tr}(\mathcal{A})$? We choose a MAD family, say of size \mathfrak{c} , $\mathcal{C} = \{c_\alpha : \alpha < \mathfrak{c}\} \subset [\omega]^\omega$, and we also fix an a.d. family $\mathcal{G}_0 = \{g_\alpha : \alpha < \mathfrak{c}\} \subset \omega^\omega$. For each $\alpha < \mathfrak{c}$, we define $h_\alpha = (f_0 \upharpoonright c_\alpha) \cup (g_\alpha \upharpoonright (\omega \setminus c_\alpha))$ and we put $\mathcal{A}_0 = \{h_\alpha : \alpha < \mathfrak{c}\}$. As \mathcal{C} and \mathcal{G}_0 are a.d., $\mathcal{A}_0 \subset \omega^\omega$ is a.d. and $f_0 \in \text{tr}(\mathcal{A}_0)$. So, now we want to continue this process and extend \mathcal{A}_0 to \mathcal{A}_α with $f_\alpha \in \mathcal{A}_\alpha$ for each $\alpha < \text{non}(\mathcal{M})$. At stage α , put $\mathcal{B} = \bigcup_{\beta < \alpha} \mathcal{A}_\beta$. If $f_\alpha \in \text{tr}(\mathcal{B})$, then there is nothing to do. Else extend $\mathcal{B} \cap f_\alpha$ to a MAD family \mathcal{C} in $[f_\alpha]^\omega$, and let $\{c_\gamma : \gamma < \mathfrak{c}\}$ enumerate $\{\text{dom}(p) : p \in \mathcal{C} \setminus (\mathcal{B} \cap f_\alpha)\}$ (we may assume this has size \mathfrak{c}). Let $\mathcal{G}_\alpha = \{g_\gamma : \gamma < \mathfrak{c}\} \subset \omega^\omega$ be an a.d. family. We would like to set $h_\gamma = (f_\alpha \upharpoonright c_\gamma) \cup (g_\gamma \upharpoonright (\omega \setminus c_\gamma))$, and put $\mathcal{A}_\alpha = \mathcal{B} \cup \{h_\gamma : \gamma < \mathfrak{c}\}$. But for \mathcal{A}_α to be an a.d. family the collection of the \mathcal{G}_α needs to satisfy some conditions. For two sets $A, B \subset \omega^\omega$, let us write $A \perp B$ to mean $\forall f \in A \forall g \in B [f \cap g < \omega]$. We need that

1. $\mathcal{G}_\alpha \subset \omega^\omega$ is an a.d. family.
2. $|\mathcal{G}_\alpha| = \mathfrak{c}$.
3. for all $\beta < \alpha < \text{non}(\mathcal{M})$, $\mathcal{G}_\alpha \perp \mathcal{G}_\beta$
4. $\mathcal{G}_\alpha \perp \{f_\beta : \beta \leq \alpha\}$.

Conditions (3) and (4) imply that there is some “thin subset” of $\omega \times \omega$ in which every function in \mathcal{G}_α is contained. The thinnest such sets are, of course, functions and more generally subsets of $\omega \times \omega$ that have finite intersection with each vertical column c_n . But if a set $S \subset \omega \times \omega$ has finite intersection with each c_n , and yet there is even an infinite a.d. family of functions $\mathcal{G} \subset \omega^\omega$ such that $\forall g \in \mathcal{G} [g \subset S]$, then it must be the case that $\lim_{n \rightarrow \infty} |S \cap c_n| = \infty$. So we are led to consider slaloms, and the slaloms must be “fat”.

DEFINITION 5.10. We say that a function $S : \omega \rightarrow [\omega]^{<\omega}$ is a *fat slalom* if for each $n \in \omega$, $|S(n)| = 2^n$.

The choice of 2^n here is arbitrary; any $h \in \omega^\omega$ with $\lim_{n \rightarrow \infty} h(n) = \infty$ would do. But the point is that if S is a fat slalom, there is an a.d. family of functions $\mathcal{G} \subset \omega^\omega$ with $|\mathcal{G}| = \mathfrak{c}$ such that $\forall g \in \mathcal{G} \forall n \in \omega [g(n) \in S(n)]$: simply view the slalom as the level of a full binary tree, and take the branches through the tree. So we would like to build the \mathcal{G}_α so that every functions in it goes through some fixed fat slalom S_α . However, in order to meet the necessary almost disjointness requirements, we need to strengthen condition (3) of Bartoszyński’s theorem to get a fat slalom that is eventually disjoint from a family of fewer than $\text{non}(\mathcal{M})$ many slaloms rather than just a function that is eventually outside them.

LEMMA 5.11 (See [24]). *Let $\langle S_\xi : \xi < \lambda \rangle$ be a family of slaloms with $\lambda < \text{non}(\mathcal{M})$. There is a fat slalom S such that*

$$\forall \xi < \lambda \forall^\infty n \in \omega [S(n) \cap S_\xi(n) = 0].$$

Using this lemma it is now possible to build a family $\langle S_\alpha : \alpha < \text{non}(\mathcal{M}) \rangle$ satisfying

- (a) S_α is a fat slalom
- (b) $\forall \beta < \alpha < \text{non}(\mathcal{M}) \forall^\infty n \in \omega [S_\alpha(n) \cap S_\beta(n) = 0]$
- (c) $\forall \beta \leq \alpha < \text{non}(\mathcal{M}) \forall^\infty n \in \omega [f_\beta(n) \notin S_\alpha(n)]$.

This is exactly what we need to finish the construction.

As a final remark, we mention that there are other ways in which cardinal invariants aid diagonalizations of length \mathfrak{c} . Some diagonalizations of length \mathfrak{c} may be performed by treating the various possible inequalities that may obtain between certain cardinal invariants as separate cases. For example, it is sometimes possible to prove the existence of a certain kind of object in ZFC by giving separate, different constructions in the three cases $\mathfrak{r} < \mathfrak{h}$, $\mathfrak{r} = \mathfrak{h}$, and $\mathfrak{r} > \mathfrak{h}$, where \mathfrak{r} and \mathfrak{h} are appropriately chosen cardinal invariants. In arguments of this sort, even though the existence of an object of the required sort is established, some of its properties, such as, perhaps, its size, are left undetermined, as they depend on which of the three cases holds. A recent example of a construction of this sort is Shelah [25] (the caveat about undetermined properties is not true here).

§6. Some Open Questions. We conclude with a discussion of some of the main open problems involving the construction of almost disjoint families of different sorts. A positive solution to any of these problems requires a diagonalization of length \mathfrak{c} to be carried out in ZFC.

An outstanding question that is still open concerns the existence of ADRs (cf. Section 4)

QUESTION 6.1. *Suppose \mathcal{I} is dense ideal in $\mathcal{P}(\omega)$. Is there a completely separable MAD family $\mathcal{A} \subset \mathcal{I}$?*

Shelah [25] has recently shown that the answer is yes if $\mathfrak{c} < \aleph_\omega$. The argument there proceeds in three cases depending on whether $\mathfrak{s} < \mathfrak{a}$, $\mathfrak{s} = \mathfrak{a}$, or $\mathfrak{s} > \mathfrak{a}$, where \mathfrak{s} is the splitting number.

Another open problem concerns the existence of indestructible MAD families (cf. Section 5). There is no forcing notion \mathbb{P} adding a new real for which we know how to construct a \mathbb{P} -indestructible MAD family (either of sets or of functions) in ZFC. A Sacks indestructible MAD family is provably the weakest such object in the following sense.

LEMMA 6.2 (See [11] and [17]). *Suppose \mathbb{P} is a forcing notion that adds a new real, and suppose \mathcal{A} is a MAD family (either in $[\omega]^\omega$ or in ω^ω). If \mathcal{A} is \mathbb{P} -indestructible, then \mathcal{A} is also Sacks indestructible.*

So the weakest sort of indestructibility we can ask for is:

QUESTION 6.3. *Is there a Sacks indestructible MAD family (either in $[\omega]^\omega$ or in ω^ω)? Is there one of size \mathfrak{c} ?*

It is not too hard to see that if $\mathfrak{a} < \mathfrak{c}$, then any MAD family of size \mathfrak{a} is Sacks indestructible. Therefore, for the first question, we may assume without loss that $\mathfrak{a} = \mathfrak{c}$. For the second question, it is known that the answer is yes if either $\mathfrak{b} = \mathfrak{c}$ or $\text{cov}(\mathcal{M}) = \mathfrak{c}$ holds.

For the case of MAD families in $[\omega]^\omega$ asking for a Cohen indestructible MAD family is the same as asking for a strongly MAD family, while for the case of ω^ω , it is the same as asking for a Van Douwen MAD $\mathcal{A} \subset \omega^\omega$ that is Cohen indestructibly Van Douwen.

QUESTION 6.4. *Is there a strongly MAD family (either in $[\omega]^\omega$ or in ω^ω)?*

It is known that the answer is yes if $\mathfrak{b} = \mathfrak{c}$. Unlike for a Sacks indestructible MAD family, it is shown in [23] that the weak Freese–Nation property of $\mathcal{P}(\omega)$ ($\text{wFN}(\mathcal{P}(\omega))$) implies that all strongly MAD families have size at most \aleph_1 . It is shown in [13] that $\text{wFN}(\mathcal{P}(\omega))$ holds in any model gotten by adding fewer than \aleph_ω Cohen reals to a ground model satisfying CH. So, in particular, it is consistent to have no strongly MAD families of size \mathfrak{c} . So if there is a ZFC construction of one, then it must leave the cardinality of the family indeterminate. More information about indestructible MAD families of sets can be found in [11].

We conclude with a discussion of a problem regarding metrizable Fréchet groups. Recall that a topological space X is *Fréchet* if whenever a point $p \in X$ is in the closure of a set $A \subset X$, there is a sequence of points A converging to p . A well-known question of Malykhin asks whether every countable Fréchet topological group is metrizable. If $\mathfrak{p} > \omega_1$, Malykhin’s question has a negative answer. Indeed, in this situation, one can put a non-metrizable Fréchet topology on the group $\langle [\omega]^{<\omega}, \Delta \rangle$, where Δ denotes symmetric difference. Given an ideal $\mathcal{I} \subset \mathcal{P}(\omega)$, let us say that a set $A \subset [\omega]^{<\omega}$ is *\mathcal{I} -positive* if for every $a \in \mathcal{I}$, $\exists s \in A [s \cap a = 0]$. Let us say that \mathcal{I} is *Fréchet* if for every \mathcal{I} -positive $A \subset [\omega]^{<\omega}$, there is an infinite $B \in [A]^\omega$ so that $\forall a \in \mathcal{I} [|a \cap (\bigcup B)| < \omega]$. If \mathcal{I} is a Fréchet ideal which is not countably generated, then we can define a non-metrizable Fréchet topology on $\langle [\omega]^{<\omega}, \Delta \rangle$ by stipulating that

$$\{A \subset [\omega]^{<\omega} : \exists a \in \mathcal{I} \forall s \in [\omega]^{<\omega} [s \cap a = 0 \implies s \in A]\}$$

is a neighborhood base at 0. If $\mathfrak{p} > \omega_1$, then *any* ideal that is \aleph_1 generated is Fréchet. The following question of Gruenhage and Szeptycki [15] asks if a Fréchet ideal that is not countably generated can be constructed in ZFC, and more specifically, whether there are any that are generated by almost disjoint families.

QUESTION 6.5. *Is there an uncountable a.d. family $\mathcal{A} \subset [\omega]^\omega$ such that $\mathcal{I}(\mathcal{A})$ is Fréchet? Is there a Fréchet ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ that is not countably generated?*

Brendle and Hrušák [10] have recently shown that it is consistent that every uncountably generated Fréchet ideal has no fewer than \mathfrak{c} generators.

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