# Quiz 5 Solutions

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**Problem 1.** Let $\mathcal{U}$ be the set of all possible arrangements of the 26 letters without any repetitions. Let $A_1$ be the set of arrangements with the word INCH. Let $A_2$ be the set of arrangements with the word LOST, and let $A_3$ be the set of arrangements with the word THIN. We want to count $\left|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3\right|$. First of all, $|\mathcal{U}| = 26!$. Now for $|A_1|$, we treat INCH as one symbol, and so we have 23 symbols to arrange. So $|A_1| = 23!$. The same for $|A_2|$ and $|A_3|$. For $|A_1 \cap A_2|$, we treat INCH as one symbol and LOST as another symbol, and so we are arranging 20 symbols. So $|A_1 \cap A_2| = 20!$. Next, for $|A_2 \cap A_3|$, since there must be no repetitions, the only possibility is for LOSTHIN to be one symbol. So again we are arranging 20 symbols, and so $|A_2 \cap A_3| = 20!$. For $|A_3 \cap A_1|$, notice that it is impossible to have both THIN and INCH occurring because some letter will be repeated. So $|A_3 \cap A_1| = 0$. It follows that $|A_1 \cap A_2 \cap A_3| = 0$ also. So the answer is $26! - 3 \times 23! + 2 \times 20!$.

**Problem 2.** Counting the number of selections a single person can make is a bit tricky because a person is allowed to choose two balloons of the same color. If $i$ is the number of color that can be used, then counting this is the same as counting the number of ways to distribute two identical objects into $i$ distinct boxes, which is $\binom{2+i-1}{2} = \binom{i+1}{2}$. Now, let $\mathcal{U}$ be the set of all possible ways for the 6 people to make their selections. For each $n$ between 1 and 7, let $A_n$ be the set of ways for the 6 people to make their selections if color $n$ is never used. We are trying to count $\left|\bar{A}_1 \cap \cdots \cap \bar{A}_7\right|$. $|\mathcal{U}| = P\left(\binom{8}{2}, 6\right)$. For each $n$, $|A_n| = P\left(\binom{7}{2}, 6\right)$, and so $S_1 = \binom{7}{1} P\left(\binom{7}{2}, 6\right)$. Given a pair $n \neq m$, $|A_n \cap A_m| = P\left(\binom{6}{2}, 6\right)$ (since now only 5 colors can be used), and so $S_2 = \binom{7}{2} P\left(\binom{6}{2}, 6\right)$. For triples $m \neq n \neq k$, $|A_n \cap A_m \cap A_k| = P\left(\binom{5}{2}, 6\right)$, and so $S_3 = \binom{7}{3} P\left(\binom{5}{2}, 6\right)$. Similarly, $S_4 = \binom{7}{4} P\left(\binom{4}{2}, 6\right)$. If 5 colors are excluded, then any given person can make only $\binom{3}{2} = 3$ selections. So it is impossible for all the selections to be different. So $S_5$ and above are 0. So the answer is $P\left(\binom{8}{2}, 6\right) - \binom{7}{1} P\left(\binom{7}{2}, 6\right) + \binom{7}{2} P\left(\binom{6}{2}, 6\right) - \binom{7}{3} P\left(\binom{5}{2}, 6\right) + \binom{7}{4} P\left(\binom{4}{2}, 6\right)$