Quiz 4 Solutions

Dilip Raghavan

November 11, 2008

Problem 1. (a) If the first slot is not a 0, then we have 3 possibilities for the first slot and a_{n-1}. If the first slot is 0, then we nust have an odd number of 0 in what follows. The total number of sequences of length $n-1$ is 4^{n-1}, and of these a_{n-1} have an even number of 0s. So there are $4^{n-1} - a_{n-1}$ possibilities for this case. So $a_n = 3a_{n-1} + 4^{n-1} - a_{n-1} = 2a_{n-1} + 4^{n-1}$.

(b) The homogeneous solution is $A2^n$. Look for a particular solution of the form $p(n) = B4^n$. Substituting, we get the equation $B4^n = 2B4^{n-1} + 4^{n-1}$, which simplifies to $4B = 2B + 1$ Hence, $B = \frac{1}{2}$. So $a_n = A2^n + \frac{1}{2}4^n$. Using the initial condition $a_1 = 3$, we get $A = \frac{1}{2}$.

Problem 2. Let a_n be the number of such subsets. We find a recurrence relation for a_n. If a subset does not contain n, then it is subset of $\{1, 2, \ldots, n-1\}$ with no consecutive integers. There are a_{n-1} such subsets. If the subset does contain n, then it cannot contain $n-1$, and the remainder of the subset must be a subset of $\{1, 2, \ldots, n-2\}$ with no consecutive integers. There are a_{n-2} such subsets. So $a_n = a_{n-1} + a_{n-2}$, which is the Fibonacci relation. The general solution is $a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$. But the initial conditions are slightly different $a_1 = 2$ and $a_2 = 3$. Using these, we can find A and B.