MAT344H1F
Mid–term Exam, October 28, 2008
Duration: 1 Hour 45 Minutes
No Aids Allowed

This exam has 8 questions, for a total of 100 points. Answer the questions in the spaces provided. If you run out of room for an answer, continue on the back of the page.

Please read the questions carefully. Be sure to justify your answer. An answer by itself will not get any credit.

Last Name: ______________________________

First Name: ______________________________

Student ID: ______________________________

Question	Points	Score
1	10	
2	10	
3	15	
4	10	
5	15	
6	15	
7	10	
8	15	
Total:	100	

1. How many ways are there to form a committee of 5 people from 10 men and 10 women if

   (a) (5 points) The committee must have *at least* three women.

   Solution: Break into cases depending on exactly how many women there are:

   $$\binom{10}{3}\binom{10}{2} + \binom{10}{4}\binom{10}{1} + \binom{10}{5}\binom{10}{0}$$

   (b) (5 points) The committee must have *at least* three women, but John and Mary cannot both be choosen.

   Solution: The same three cases. But now subtract the selections that have both John and Mary:

   $$\left[\binom{10}{3}\binom{10}{2} - \binom{9}{2}\binom{9}{1}\right] + \left[\binom{10}{4}\binom{10}{1} - \binom{9}{3}\binom{9}{0}\right] + \left[\binom{10}{5}\binom{10}{0} - 0\right]$$

2. (10 points) How many arrangements of the letters in STATISTICALLY have *all* of the following properties:

   (i) at least two consonants between successive vowels

   (ii) the arrangement starts with a vowel

   (iii) the consonants are in alphabetical order.

Solution: First arrange the vowels: $\frac{4!}{2!2!}$ ways. Now distribute empty slots for the consonants making sure that there are at least 2 slots between any two vowels and making sure that the space before the first vowel is empty: $\binom{3+4-1}{3}$ ways. Now there is just one way to arrange the consonants in alphabetical order. So the answer is $\dfrac{4!}{2!2!} \times \dbinom{3+4-1}{3}$

3. (15 points) How many ways are there to arrange 8 identical red flags, 9 identical blue flags, and 10 different, distinct national flags onto 11 distinct flagpoles if the order of the flags from the ground up matters?

Solution: First distribute empty slots for the flags: $\binom{27+11-1}{27}$ ways. Next arrange the flags in the slots. Since the blue flags are all identical to each other and the red flags are all identical to each other, there are $\frac{27!}{8!9!}$ ways to do this. So the answer is $\binom{27+11-1}{27} \times \frac{27!}{8!9!}$

4. (10 points) It is Valentine's Day and you want to buy some flowers. You go to a flower shop which has an unlimited supply of red roses, pink roses, yellow roses and white roses. In how many ways can you select a bouquet of 12 roses if you want to have *at most* five roses of any one color?

Solution: The ordinary generating function is $\left(x^0 + x^1 + x^2 + x^3 + x^4 + x^5\right)^4$. You are looking for the co-efficient of x^{12}.

5. (15 points) How many ways are there to distribute the 16 integers from 1 to 16 (inclusive) into three distinct boxes: Box A, Box B and Box C if Boxes A and B must each contain *at least* two integers and Box C must contain an even number of integers?

Solution: The exponential generating function is

$$\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots\right)^2\left(\frac{x^0}{0!}+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)=(e^x-1-x)^2\left(\frac{e^x+e^{-x}}{2}\right).$$

You are looking for the co-efficient of $\frac{x^{16}}{16!}$

6. (15 points) Give a combinatorial argument to prove

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \binom{2n}{n+1}$$

[A *combinatorial argument* means that you must describe a counting problem and compute the answer to this counting problem in two different ways to get the left and right hand sides of the identity.]

Solution: Suppose you have n distinct apples and n distinct oranges and you want to select $n+1$ fruits out of this. There are $\binom{2n}{n+1}$ ways to do this. Now count the same thing by breaking into cases depending on exactly how many apples are choosen. Note that we must choose at least one apple because we only have n oranges and we want to select $n+1$ pieces of fruit. Now, if we decide to choose exactly k apples, there are $\binom{n}{k}$ ways to choose the apples. We must choose $n+1-k$ oranges and there are $\binom{n}{n+1-k}$ ways to do this. But since $\binom{n}{i}$ is always equal to $\binom{n}{n-i}$, $\binom{n}{n+1-k} = \binom{n}{n-n-1+k} = \binom{n}{k-1}$. So there are $\binom{n}{k} \times \binom{n}{k-1}$ ways to choose the apples and the oranges. k runs from 1 to n since we must have at least one apple. So we get

$$\sum_{k=1}^{n} \binom{n}{k} \times \binom{n}{k-1}$$

7. (10 points) Find a recurrence relation for a_n, the number of ways to place the following three types of flags on a flagpole n feet tall: red flags which are 1 foot tall, blue flags which are 1 foot tall, and yellow flags which are 2 feet tall, subject to the requirement that a red flag cannot be immediately followed by a yellow flag (flags of the same color are considered identical). Be sure to specify what the initial conditions are.

Solution:

- Case I: The arrangement begins with a red flag. We have a total of a_{n-1} possibilities for the remaining pole that is $n-1$ feet tall. Of these we must exclude the ones starting with a yellow flag. So there are $a_{n-1} - a_{n-3}$ ways.
- Case II: The arrangement begins with a blue flag. a_{n-1} ways.
- Case III: The arrangement begins with a yellow flag. a_{n-2} ways.

So $a_n = a_{n-1} - a_{n-3} + a_{n-1} + a_{n-2} = 2a_{n-1} + a_{n-2} - a_{n-3}$. $a_1 = 2$, $a_2 = 5$. $a_3 = 11$

8. (15 points) Suppose a coin is flipped 21 times in a row and 6 heads and 15 tails are obtained. How many such sequences of heads and tails are there which have *all* of the following properties:

(i) the sequence begins and ends with a tail

(ii) there are no consecutive heads

(iii) *any* run of tails has odd length.

A run of tails is a sequence of one or more consecutive tails. For example, the sequence $\text{HH}\underline{\text{TTTT}}\text{H}\underline{\text{T}}\text{H}\underline{\text{TT}}\text{HHH}\underline{\text{T}}$ has four runs of tails (the underlined portions).

Solution: First arrange the 6 heads:

H H H H H H

We must distribute the tails in the boxes before, in between, and after the heads. Conditions (i)–(iii) imply that all of these boxes must be non–empty, and have an odd number of tails. So the ordinary generating function for the distribution is

$$(x^1 + x^3 + x^5 + x^7 + \cdots)^7 = \left(\frac{x}{1-x}\right)^7$$

We want the co-efficient of x^{15}.

End of exam.