HW9 Solutions

Dilip Raghavan

November 17, 2008

1 Section 7.5

Problem 8. If the first slot is 1 or 2, then there are a_{n-1} possibilities for the rest. If the first slot is 0, then out of the total a_{n-1} possibilities, we need to exclude the ones starting with 12. There are a_{n-3} such sequences. So $a_n = 2a_{n-1} + a_{n-1} - a_{n-3} = 3a_{n-1} - a_{n-3}$. Put $g(x) = \sum_{n=0}^{\infty} a_n x^n$. Note that $a_1 = a_2 = 0$ (since we must have the pattern 012), and that $a_3 = 1$. From these we can calculate $a_0 = -1$. Using the recurrence relation and the inital conditions, we can get $g(x) = \dfrac{1 + 3x}{1 - 3x + x^3}$.

Problem 13. Let a_n be the number of n–digit quaternary sequences with an odd number of 1s and an odd number of 2s. Let b_n be the number of n–digit quaternary sequences with an even number of 1s and an odd number of 2s, and let c_n be the number of n–digit quaternary sequences with an odd number of 1s and an even number of 2s. Note that number of n–digit quaternary sequences with an even number of 1s and an even number of 2s is $4^n - a_n - b_n - c_n$. So we have the recurrence

$$\begin{aligned} a_n &= 2a_{n-1} + b_{n-1} + c_{n-1} \\ b_n &= b_{n-1} + 4^{n-1} - c_{n-1} \\ c_n &= c_{n-1} + 4^{n-1} - b_{n-1} \end{aligned}$$

Solving these is very similar to example 5 in the book and done in class.

Problem 14. Let a_n be the number of n–digit ternary sequences whose sum of digits is 0 mod 3. Let b_n be the number of n–digit ternary sequences whose sum of digits is 1 mod 3. Note that the number of n–digit ternary sequences whose sum of digits is 2 mod 3 is $3^n - a_n - b_n$. If the first slot is 0, then we have a_{n-1} possibilities for the remaining (since those digits must sum to 0 mod 3). If the first slot is 1, then we have $3^{n-1} - a_{n-1} - b_{n-1}$ possibilities (because the remaining digits must sum to 2 mod 3). If the first slot is 2, then we have b_{n-1} possibilities (because the remaining digits must sum to 1 mod 3). So $a_n = a_{n-1} + 3^{n-1} - a_{n-1} - b_{n-1} + b_{n-1} = 3^{n-1}$. So we just get $a_n = 3^{n-1}$, and there is no need to solve anything!!

2 Section 8.1

Problem 23. Let the universe $\mathcal{U}$ be the set of all possible 10 person committees. This is an intersection problem, since we want at least one person from *each* of the three groups. So let A_1 be the set of 10 person committees with *no* mathematicians; let A_2 be the set of 10 person committees with *no* statisticians; and let A_3 be the set of 10 person committees with *no* operations researchers. We want $\left|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3\right|$. First of all, $|\mathcal{U}| = \binom{37}{10}$. Next, $|A_1| = \binom{22}{10}$, $|A_2| = \binom{25}{10}$ and $|A_3| = \binom{27}{10}$. Now, $|A_1 \cap A_2| = \binom{10}{10}$, $|A_2 \cap A_3| = \binom{15}{10}$, and $|A_3 \cap A_1| = \binom{12}{10}$. Finally, $|A_1 \cap A_2 \cap A_3| = 0$. So the answer is $\binom{37}{10} - \binom{22}{10} - \binom{25}{10} - \binom{27}{10} + \binom{10}{10} + \binom{15}{10} + \binom{12}{10} - 0$.

Problem 28. We have 2 Ms, 2 As, 2 Ts, 1 H, 1 E, 1 I, 1 C, and 1 S. This is a union problem, since we want *one* out of many possible favorable outcomes to be true. Let $\mathcal{U}$ be the set of all possible arrangements. Let A_1 be all arrangements with both Ts before both As. Let A_2 be all arrangements with both As before both Ms. Let A_3 be all arrangements with both Ms before the E. We want $|A_1 \cup A_2 \cup A_3|$. $|A_1| = \binom{11}{4} \times \frac{7!}{2!}$ – first choose the 4 slots for the 2 Ts and 2 As and place them in those slots with the Ts coming before the As, and then arrange the rest of the letters. Similarly, $|A_2| = \binom{11}{4} \times \frac{7!}{2!}$, and $|A_3| = \binom{11}{3} \times \frac{8!}{2!2!}$. Next, $|A_1 \cap A_2| = \binom{11}{6} \times 5!$ – first choose 6 slots for the 2 Ms, 2 As, and 2 Ts, and place them in the slots, with the Ts coming first, then the As and then the Ms, and then arrange the remaining letters. Similarly, $|A_2 \cap A_3| = \binom{11}{5} \times \frac{6!}{2!}$, and $|A_3 \cap A_1| = \binom{11}{4} \times \binom{7}{3} \times 4!$. Finally, $|A_1 \cap A_2 \cap A_3| = \binom{11}{7} \times 4!$. So the answer is $\binom{11}{4} \times \frac{7!}{2!} + \binom{11}{4} \times \frac{7!}{2!} + \binom{11}{3} \times \frac{8!}{2!2!} - \binom{11}{6} \times 5! - \binom{11}{5} \times \frac{6!}{2!} - \binom{11}{4} \times \binom{7}{3} \times 4! + \binom{11}{7} \times 4!$.

Problem 36. Let $\mathcal{U}$ be the class of 30 children. Let L be the set of students taking Latin, let G be the set of children taking Greek, and let H be the set of those taking Hebrew. We know that 8 children take no language. So $8 = \left|\bar{L} \cap \bar{G} \cap \bar{H}\right| = |\mathcal{U}| - |L| - |G| - |H| + |L \cap G| + |G \cap H| + |H \cap L| - |L \cap G \cap H| = 30 - 20 - 14 - 10 + |L \cap G| + |G \cap H| + |H \cap L| - 0$. Therefore, $|L \cap G| + |G \cap H| + |H \cap L| = 22$. Since there are 8 students who are taking no language, it follows that $|L \cap G| + |G \cap H| + |H \cap L|$ is *precisely equal* to $|L \cup G \cup H|$. But since the sets $L \cap G$, $G \cap H$ and $L \cap G$ are all pairwise disjoint (i.e. the intersection of any two of them is disjoint because there are 0 students taking all three languages), it follows that

$$20 = |L| = |L \cap G| + |L \cap H|$$
$$14 = |G| = |G \cap L| + |G \cap H|$$
$$10 = |H| = |H \cap L| + |H \cap G|$$

These are 3 equations in 3 variables. Solving simultaneously, we get $|L \cap G| = 12$, $|L \cap H| = 8$, $|H \cap G| = 2$.